

THE GENERALIZATION OF ANALYTIC
FUNCTIONS
ON THE THEORY OF WAVES AND GREEN'S
METHOD *

THE GENERALIZATION OF ANALYTIC
FUNCTIONS†

INTRODUCTION

THE generalization which is treated in the following pages has already been the subject of several investigations of mine, in the first place in several notes, published in the "Rendiconti" of the Reale Accademia dei Lincei, then in an extended memoir which appeared in the "Acta Mathematica." Several of the lectures which I read at Stockholm were also devoted to this subject. And it is now my purpose, in returning to it, to consider the general case in some detail, beginning with the first foundations. In treating the general case it is necessary to consider certain elements, which I have called functions of hyperspaces, and which represent extensions of the functions of curves that I have already treated several times, in particular, in a recent course at the Sorbonne.

A space of n dimensions contains spaces of $0, 1, 2, \dots, n-1$ dimensions, and for that reason we consider functions of

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†Translated from the Italian by Professor Griffith Conrad Evans, of the Rice Institute.

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these spaces. We shall begin by extending to these functions the fundamental concepts of continuity and differentiation, and we shall consider the condition that a function be of the first degree. This condition depends upon an extension of Stokes's theorem. We shall then consider a relation between these functions analogous to that of monogeneity, which for functions in the ordinary sense was established by Cauchy. This leads to new types of equations with functional derivatives, which present analogies with the equation of Laplace.

We can separate the functions with which we are dealing into elementary and otherwise. The former have interesting properties and applications. A certain operation of composition turns out to possess quite curious arithmetical properties.

We shall finally develop the operations of differentiation and integration, and the extension of Cauchy's theorem in complete generality.

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First Lecture

GENERAL OBSERVATIONS ON HYPERSPACES — GENERAL FORMULÆ FOR MATRICES, AND RELATIONS BETWEEN THE DIRECTION COSINES OF HYPERSPACES — FUNCTIONS OF HYPERSPACES AND THEIR DERIVATIVES — EXTENSION OF STOKES'S THEOREM — CONDITIONS WHICH THE DERIVATIVES OF FUNCTIONS OF HYPERSPACES MUST SATISFY, AND FORMULÆ FOR THE TRANSFORMATION OF COÖRDINATES — ISOGENEITY — CONDITIONS FOR ISOGENEITY.

I. General observations on hyperspaces

I. A hyperspace (space of n dimensions) will be characterized by the multiplicity of values of n independent variables x_1, x_2, \dots, x_n . A hyperspace S_r of r dimensions ($r < n$), contained in it, will correspond to the multiplicity of values which the x_1, x_2, \dots, x_n assume when they are constrained

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by $n - r$ independent relations, or in other words, when they depend on r independent variables $\omega_1, \omega_2, \dots, \omega_r$ to which they are bound by the n relations

$$(1) \quad \begin{cases} x_1 = x_1(\omega_1, \omega_2, \dots, \omega_n) \\ x_2 = x_2(\omega_1, \omega_2, \dots, \omega_r) \\ \vdots \\ x_n = x_n(\omega_1, \omega_2, \dots, \omega_r) \end{cases}$$

We assume the differentiability of the preceding relations, and obtain

$$(2) \quad dx_i = \sum_{s=1}^r \frac{\partial x_i}{\partial \omega_s} d\omega_s \quad (i = 1, 2, \dots, n).$$

2. Let us consider the matrix

$$(3) \quad \begin{vmatrix} \frac{\partial x_1}{\partial \omega_1} & \frac{\partial x_2}{\partial \omega_1} & \dots & \frac{\partial x_n}{\partial \omega_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial \omega_r} & \frac{\partial x_2}{\partial \omega_r} & \dots & \frac{\partial x_n}{\partial \omega_r} \end{vmatrix}$$

Let Δ^2 be the square of this matrix, and let us assume that if the sign of Δ is given at one point, it is fixed by continuity at all other points. When the sign of Δ is given we shall say that the *direction* of the hyperspace S_r is fixed. The quantity

$$dS_r = \Delta d\omega_1 d\omega_2 \dots d\omega_r$$

will be called the *element of the hyperspace*.

Let us take a minor determinant of the matrix (3)

$$\Delta_{i_1 i_2 \dots i_r} = \begin{vmatrix} \frac{\partial x_{i_1}}{\partial \omega_1} & \frac{\partial x_{i_2}}{\partial \omega_1} & \dots & \frac{\partial x_{i_r}}{\partial \omega_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{i_1}}{\partial \omega_r} & \frac{\partial x_{i_2}}{\partial \omega_r} & \dots & \frac{\partial x_{i_r}}{\partial \omega_r} \end{vmatrix}$$

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and write

$$(4) \quad \alpha_{i_1 i_2 \dots i_r} = \frac{\Delta_{i_1 i_2 \dots i_r}}{\Delta}$$

The $\alpha_{i_1 i_2 \dots i_r}$ will not change if we substitute for the $\omega_1, \omega_2, \dots, \omega_r$ other variables bound by arbitrary relations to the first, and their signs will change only if we change the sign of the hyperspace; we shall call them the *direction cosines* of the hyperspace. We see at once that they must satisfy the relation

$$(A) \quad \sum_i \alpha_{i i_1 i_2 \dots i_r}^2 = 1,$$

in which \sum_i denotes summation extended over all the combinations of the indices i_1, i_2, i_n .

3. If a space S_{n-r} has a point in common with S_r , and the direction cosines of S_{n-r} are denoted by $\beta_{h_1 \dots h_{n-r}}$, we shall say that the two hyperspaces are normal to each other when we have the relation

$$\alpha_{i_1 i_2 \dots i_r} = \beta_{h_1 h_2 \dots h_{n-r}},$$

where all the i 's are different from the h 's, and the series of numbers $i_1 i_2, \dots, i_r, h_1, h_2, \dots, h_{n-r}$ is a permutation of the numbers $1, 2, \dots, n$, which is always odd or always even.

4. Whatever l may be, we can write

$$(5) \quad d\omega_l = \sum_i A_{i i_1 i_2 \dots i_{r-1}} \frac{d(x_{i_1}, x_{i_2} \dots x_{i_{r-1}})}{d(\omega_1, \omega_2 \dots \omega_{l-1} \omega_{l+1} \dots \omega_r)} \quad (l = 1, 2, \dots, r)$$

in which the sum is extended over all the combinations of the indices $i_1, i_2 \dots i_{r-1}$, and the A 's are certain, in part indeterminate, infinitesimal parameters. In fact if we form the matrix of the coefficients of the A 's, among its minors will be found the $r - 1$ th powers of the minors of the matrix (3), and so not all the minors of that matrix can be zero. If we substitute the values (5) in

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the equations (2) we obtain

$$(6) \quad dx_s = -\sum_i \frac{d(x_s, x_{i_1} \cdots x_{i_{r-1}})}{d(\omega_1, \omega_2 \cdots \omega_r)} A_{i_1 i_2 \cdots i_{r-1}}.$$

Hence if $a_{i_1, i_2 \cdots i_{r-1}} = -\Delta A_{i_1 i_2 \cdots i_{r-1}}$ we shall have

$$(7) \quad dx_s = \sum_i a_{i_1 i_2 \cdots i_{r-1}} \alpha_{s i_1 i_2 \cdots i_{r-1}}.$$

5. Besides the equations (A) the α satisfy other relations, which we shall find in the next section.

2. General formulæ about matrices. Relations between the direction cosines of a hyperspace

1. We shall establish in this section several fundamental formulæ regarding the minors of matrices, which we shall often have occasion to use. Let us consider the two matrices

$$(1) \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{vmatrix} \quad (2) \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{vmatrix}$$

the first with r rows, and the second with p rows, ($n > r \geq p$), both however with the same elements. Let us write

$$\begin{vmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ri_1} & a_{ri_2} & \cdots & a_{ri_r} \end{vmatrix} = A_{i_1 i_2 \cdots i_r}, \quad \begin{vmatrix} a_{1h_1} & a_{1h_2} & \cdots & a_{1h_p} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ph_1} & a_{ph_2} & \cdots & a_{ph_p} \end{vmatrix} = B_{h_1 h_2 \cdots h_p}$$

and consider

$$\Delta_s = \begin{vmatrix} a_{si_1} & a_{si_2} & \cdots & a_{si_{r+1}} \\ a_{1i_1} & a_{1i_2} & \cdots & a_{1i_{r+1}} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ri_1} & a_{ri_2} & \cdots & a_{ri_{r+1}} \end{vmatrix} = 0 \quad (s = 1, 2, \cdots p).$$

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We shall have

$$\begin{aligned}
 0 &= \sum_{1^p} \frac{\partial B_{h_1 h_2 \dots h_p}}{\partial a_{s h_1}} \Delta_s \\
 &= - \sum_{1^p} \frac{\partial B_{h_1 h_2 \dots h_p}}{\partial a_{s h_1}} \sum_{1^{r+1}}^t (-1)^t a_{s t_1} A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+1}} \\
 &= - \sum_{1^t}^{r+1} A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+1}} \sum_{1^p} (-1)^s a_{s t_1} \frac{\partial B_{h_1 \dots h_p}}{\partial a_{s h_1}}.
 \end{aligned}$$

From this it follows that

$$(3) \quad \sum_{1^t}^{r+1} (-1)^t A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+1}} B_{t_1 h_2 \dots h_p} = 0.$$

2. This is the formula which we wished to obtain. In particular, if we take as identical the two matrices (1) and (2), we shall have

$$(3') \quad \sum_{1^t}^{r+1} (-1)^t A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+1}} A_{t_1 h_2 \dots h_r} = 0.*$$

Among these equations let us notice specially the following, from which the others all follow :

$$\begin{aligned}
 (4) \quad 0 &= A_{t_1 t_2 h_1 \dots h_{r-2}} A_{t_3 t_4 h_1 \dots h_{r-2}} + A_{t_1 t_2 h_1 \dots h_{r-2}} A_{t_3 t_4 h_1 \dots h_{r-2}} \\
 &\quad + A_{t_1 t_4 h_1 \dots h_{r-2}} A_{t_2 t_3 h_1 \dots h_{r-2}}.^\dagger
 \end{aligned}$$

3. From the preceding formulæ we see that the direction cosines of a hyperspace must satisfy the relations

$$(B) \quad \sum_{1^s}^{r+1} (-1)^s \alpha_{t_1 t_2 \dots t_{s-1} t_{s+1} \dots t_{r+1}} \alpha_{t_s h_2 \dots h_r} = 0.$$

* Vedi Antonelli: "Nota sulle relazioni indipendenti" ecc. ("Ann. d. Scuola Normale sup. di Pisa," Vol. III, page 71 e seg.)

† *Ibid.*, page 73.

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3. *Functions of hyperspaces and their derivatives**

1. A variable ϕ will be said to be a *function of the hyperspace* S_r (of r dimensions) or a function of *order* r , if to every possible hyperspace with fixed direction corresponds a value of ϕ . This correspondence will be denoted by means of the symbol $\phi = \phi | [S_r] |$. We shall assume that we are dealing only with closed hyperspaces S_r .†

Let us take a point P of S_r and through it draw a hyperspace S_{n-r} normal to S_r , taking in S_{n-r} a small neighborhood s of P . If we make P describe all the points of S_r we shall generate a portion of n -dimensional space, which we shall call a neighborhood of S_r . While P is describing S_r any other point P' of s describes a new hyperspace S'_r , which we shall say belongs to the neighborhood of S_r . The function $\phi | [S_r] |$ will be said to be continuous if, when we take a quantity σ arbitrarily small, we can find a neighborhood of S_r such that

$$\text{mod } [\phi | [S'_r] | - \phi | [S_r] |] < \sigma,$$

where S'_r belongs to that neighborhood.

Besides the continuity of $\phi | [S_r] |$ let us admit also the following property. Let us pass from the hyperspace S_r to the hyperspace S'_r by giving to each point of S_r a displacement ϵ which varies continuously from point to point. The displacement ϵ generates a hyperspace S_{r+1} of $r+1$ dimensions, of amplitude say, σ . We shall assume that we can make $\{\phi | [S'_r] | - \phi | [S_r] | \}$ less than a number chosen arbitrarily small, provided σ be less than some value σ_0 .

2. With this understood, take in S_r a neighborhood s of a point P , and give to s a displacement δx_i parallel to x_i .

* Vedi la mia Nota I: "Sulle funzioni dipendenti da linee." ("Atti d. R. Acc. d. Lincei," Vol. III, fasc. 9.)

† Vedi: Betti: "Sopra gli spazii di un numero qualunque di dimensioni." (Annali di Mat., T. IV.)

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Let us denote by $\delta\phi$ the corresponding variation of ϕ , and let us suppose that the value

$$\lim_{\substack{S=0 \\ \delta x_i=0}} \frac{\partial \phi}{S \cdot \partial x_i} = \phi'_{x_i} \quad (i = 1, 2, \dots, n)$$

exists. We shall call this *the derivative of ϕ with respect to x_i at the point P* . With the assumption that the ratio which appears in the left-hand member approaches its limit uniformly, with respect to all possible points P and hyperspaces S_r , and that this limit is continuous, we can easily verify the fact that if we give to every point of S_r a displacement of components $\delta x_1, \delta x_2, \dots, \delta x_n$, the corresponding variation of ϕ is given, except for infinitesimals of higher order, by the formula

$$(1) \quad \delta\phi = \int_{S_r} \sum_i^n \phi'_{x_i} \delta x_i dS_r.$$

3. Let us find out now what conditions the ϕ'_{x_i} must satisfy. If the displacements are such as to carry the space S_r into itself, the quantity $\delta\phi$ must vanish. Hence we must have $\delta\phi = 0$ if we take (see § 1, form 7)

$$\delta x_i = \sum_h a_{h_1 h_2 \dots h_{r-1}} \alpha_{i h_1 \dots h_{r-1}},$$

whatever the quantities a may be. Hence

$$0 = \int_{S_r} \sum_h \alpha_{h_1 h_2 \dots h_{r-1}} \sum_i^n \phi'_{x_i} \alpha_{i h_1 \dots h_{r-1}} dS_r$$

and from this we have

$$(2) \quad \sum_i^n \phi'_{x_i} \alpha_{i h_1 \dots h_{r-1}} = 0$$

for every possible combination of the indices $h_2 \dots h_{r-1}$.

4. Since now the α satisfy the relations § 2, (B), we have

$$\sum_{i=1}^{r+1} (-1)^i \alpha_{q_i h_1 \dots h_{r-1}} \alpha_{q_1 \dots q_{i-1} q_{i+1} \dots q_{r+1}} = 0.$$

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If we multiply this by an undetermined parameter $\lambda_{q_1 q_2 \dots q_{r+1}}$ which satisfies the condition that it changes sign for every transposition of the indices, we shall have

$$\begin{aligned} 0 &= \sum_q \lambda_{q_1 q_2 \dots q_{r+1}} \sum_{i=1}^{r+1} (-1)^i \alpha_{q_i h_1 \dots h_{r-1}} \alpha_{q_1 \dots q_{i-1} q_{i+1} \dots q_{r+1}} \\ &= \sum_{i=1}^n \sum_q \lambda_{i q_1 \dots q_r} \alpha_{q_1 \dots q_r} \alpha_{i h_1 \dots h_{r-1}} \end{aligned}$$

and subtracting this from equation (2),

$$0 = \sum_{i=1}^n \left\{ \phi'_{x_i} - \sum_q \lambda_{i q_1 \dots q_r} \alpha_{q_1 q_2 \dots q_r} \right\} \alpha_{i h_1 \dots h_{r-1}}$$

whence

$$(3) \quad \phi'_{x_i} = \sum_q \lambda_{i q_1 \dots q_r} \alpha_{q_1 q_2 \dots q_r}^* *$$

From this it follows that

$$\begin{aligned} \delta\phi &= \int_{S_r} \sum_{i=1}^n \sum_q \lambda_{i q_1 \dots q_r} \alpha_{q_1 q_2 \dots q_r} \delta x_i dS_r \\ &= \int_{S_r} \sum_q \lambda_{q_1 q_2 \dots q_{r+1}} \left\{ \sum_{i=1}^{r+1} (-1)^{i-1} \alpha_{q_1 q_2 \dots q_{i-1} q_{i+1} \dots q_{r+1}} \delta x_{q_i} \right\} dS_r. \end{aligned}$$

Consider now the elements dS_r and suppose drawn through every point of it a segment of components $\delta x_1 \dots \delta x_n$. The locus of these segments will be a space S_{r+1} of $r+1$ dimensions. If the equations of the hyperspace S_r are

$$x_i = x_i(\omega_1, \omega_2, \dots, \omega_r) \quad (i = 1, 2, \dots, n)$$

the equations of the hyperspace S_{r+1} will be

$$x_i = x_i(\omega_1, \omega_2, \dots, \omega_r) + \omega_{r+1} \delta x_i \quad (i = 1, 2, \dots, n).$$

* See my Note II: "Sulle funzioni dipendenti da linee." ("Atti della R. Acc. dei Lincei," Vol. III, fasc. 10.)

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Let us form the matrix

$$(4) \quad \begin{vmatrix} \frac{\partial x_t}{\partial \omega_1}, & \frac{\partial x_2}{\partial \omega_1} & \dots & \frac{\partial x_n}{\partial \omega_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_1}{\partial \omega_r}, & \frac{\partial x_2}{\partial \omega_r} & \dots & \frac{\partial x_n}{\partial \omega_r} \\ \partial x_1, & \partial x_2 & \dots & \partial x_n \end{vmatrix}$$

Let us denote its square by Δ_{r+1}^2 , and the square of the matrix obtained from it by taking away the last line by Δ_r^2 . We shall have

$$\Delta_{r+1}^2 = \Delta_r^2 \left\{ \sum_q \sum_1^{r+1} (-1)^{t-1} \alpha_{q_1 \dots q_{t-1} q_{t+1} \dots q_{r+1}} \delta x_{q_t} \right\}^2$$

We can fix the direction of S_{r+1} with respect to S_r in such a way that

$$\Delta_{r+1} = (-1)^r \Delta_r \sqrt{\sum_q \sum_1^{r+1} (-1)^{t-1} \alpha_{q_1 \dots q_{t-1} q_{t+1} \dots q_{r+1}} \delta x_{q_t}}$$

where the sign of the radical is taken as positive. If now we denote the direction cosines of S_{r+1} by $\beta_{q_1 q_2 \dots q_{r+1}}$, which are calculated from the matrix (4), we shall have finally

$$\delta \phi = \int_{S_{r+1}} \sum_q \lambda_{q_1 q_2 \dots q_{r+1}} \beta_{q_1 q_2 \dots q_{r+1}} dS_{r+1}.$$

Hence if S_r is a movable hyperspace which passes from S'_r to S''_r , thus generating a S_{r+1} , we shall have

$$(5) \quad \phi |[S''_r] - \phi |[S'_r]| = \int_{S_{r+1}} \sum_q \lambda_{q_1 q_2 \dots q_{r+1}} \beta_{q_1 q_2 \dots q_{r+1}} dS_{r+1}.$$

It is well to note explicitly that besides varying from point to point of the total hyperspace (of n dimensions), the parameters λ may also vary for one and the same point according to the hyperspace to which they refer, and even

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for the same hyperspace one set of λ 's may be substituted for another provided the relations (3) are always satisfied.

5. A function $\phi | [S_r]$ will be said to be *regular* (or *simple*) when the following condition is satisfied. Let S'_r and S''_r be two hyperspaces having a common portion s , whose direction is different according as it is considered as belonging to the first or the second hyperspace. Denote by S'''_r the hyperspace which we get by taking away s from the combination of S'_r and S''_r and fix as its direction the direction of those two hyperspaces. We impose the condition

$$\phi | [S'''_r] = \phi | [S'_r] + \phi | [S''_r].$$

When ϕ is regular it follows immediately that if S_r decreases indefinitely in amplitude

$$(C) \quad \lim \phi | [S_r] = 0.$$

We have then immediately the further property that if S_r and S'_r are two hyperspaces with a common point P , whose elements at P are contained in a single S_{r+1} , of $r+1$ dimensions,

$$(6) \quad \sum_a (\lambda_{a_1 a_2 \dots a_{r+1}} - \lambda'_{a_1 a_2 \dots a_{r+1}}) \beta_{a_1 a_2 \dots a_{r+1}} = 0$$

where λ and λ' are the parameters which correspond to $\phi | [S_r]$ and $\phi | [S'_r]$ at the point P , and the β 's are the direction cosines of S_{r+1} .

Upon this basis let us consider a hyperspace S_r passing through the point P , whose element at P is defined by the equations

$$dx_i = \sum_s^r a_{is} d\omega_s \quad (i = 1, 2, \dots, n)$$

and let $S_r^{(i_1 \dots i_r) (h_1 \dots h_p)}$ denote hyperspaces passing through P defined by the equations

$$\begin{aligned} dx_{i_s} &= a_{i_s s} d\omega_s + \sum_t^p a_{i_s h_t} d\omega_{h_t} & \begin{cases} s = 1, 2, \dots, r \\ s \neq h_1, h_2, \dots, h_p. \end{cases} \\ dx_{i_v} &= \sum_t^p a_{i_v h_t} d\omega_{h_t} & (v = h_1, h_2, \dots, h_p, r+1, \dots, n.) \end{aligned}$$

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In particular let us consider the hyperspaces $S^{(i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1})}$ and $S_r^{(i_1 \dots i_{r-1} i_{r+1})}$ whose elements at P are contained in a hyperspace of $r+1$ dimensions, of which the direction cosines β are zero, except $\beta_{i_1 i_2 \dots i_{r+1}} = 1$. By means of (6), we have

$$\lambda_{i_1 i_2 \dots i_{r+1}}^{(i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1})} = \lambda_{i_1 i_2 \dots i_{r+1}}^{(i_1 \dots i_{t-1} i_{t+1} \dots i_{r+1})}$$

where the indices $i_1 i_2 \dots i_r$ denote the parameters λ corresponding to the hyperspace $S_r^{(i_1 \dots i_n)}$. Therefore we can suppress the indices and write simply

$$(7) \quad \lambda_{i_1 i_2 \dots i_{r+1}}^{(i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1})} = \Lambda_{i_1 i_2 \dots i_{r+1}}.$$

6. Two hyperspaces $S_r^{(i_1 \dots i_r)(h_1 \dots h_{p-1})}$ and $S_r^{(i_1 \dots i_r)(h_1 \dots h_p)}$ have elements at P which are contained in a S_{r+1} , whose element at P is defined by the equations

$$\begin{aligned} dx_{i_s} &= a_{i_s s} d\omega_s + \sum_{i=1}^p a_{i_s h_i} d\omega_{h_i} & \begin{cases} s = 1, 2, \dots, r \\ s \neq h_1, h_2, \dots, h_p \end{cases} \\ dx_{i_{h_p}} &= a_{i_{h_p} h_p} d\omega_{r+1} + \sum_{i=1}^p a_{i_{h_p} h_i} d\omega_{h_i} \\ dx_{i_v} &= \sum_{i=1}^p a_{i_v h_i} d\omega_{h_i}, & (v = h_1, h_2, \dots, h_{p-1}, r+1, \dots, n.) \end{aligned}$$

Hence, if we denote by β the direction cosines of S_{r+1} and by $\alpha^{(i_1 \dots i_r)(h_1 \dots h_p)}$ the direction cosines of $S_r^{(i_1 \dots i_r)(h_1 \dots h_p)}$, we shall have

$$\frac{\beta_{i_{h_p} m_1 m_2 \dots m_r}}{\alpha_{m_1 m_2 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)}} = \kappa,$$

where κ is independent of the indices m_1, m_2, \dots, m_r , and all the β 's are zero, in the indices of which i_{h_p} is missing. From this it follows by reason of (6) that

$$\sum_m \left(\lambda_{i_{h_p} m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)} - \lambda_{i_{h_p} m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_{p-1})} \right) \alpha_{m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)} = 0,$$

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where the index $(i_1 \dots i_r)(h_1 \dots h_p)$, affixed to the λ , means that refers to the hyperspace having the same index. We have then

$$\sum_m \lambda_{i_{h_p} m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)} \alpha_{m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)} = \sum_m \lambda_{i_{h_p} m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_{p-1})} \alpha_{m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)},$$

in which, by means of (3), we can substitute for the $\lambda_{i_{h_p} m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_p)}$ the $\lambda_{i_{h_p} m_1 \dots m_r}^{(i_1 \dots i_r)(h_1 \dots h_{p-1})}$, and consequently, the $\Lambda_{i_{h_p} m_1 \dots m_r}$ of formula (7).

We observe however that the hyperspace $S_r^{(i_1 \dots i_r)(h_1 \dots h_p)}$ is nothing but the hyperspace S_r , and therefore we can take for the λ 's belonging to this space, at the point P , the λ 's without index of formula (6). We have then the theorem

If ϕ is a regular function of the hyperspace S_r , contained in a hyperspace S_n , there exist for every point of S_n a system of values which can be considered as the parameters $\lambda_{i_1 i_2 \dots i_{r+1}}$ for all the hyperspaces S_r which pass through that point.

7. From the equations (5) (C), assuming that $\phi | [S_r]$ is regular we get,

$$(5') \quad \phi | [S_r] = \int_{S_{r+1}} \sum_q \Lambda_{q_1 q_2 \dots q_{r+1}} \beta_{q_1 q_2 \dots q_{r+1}} dS_{r+1}.$$

Here S_{r+1} is an arbitrary hyperspace of $r+1$ dimensions, whose boundary is S_r . If S_{r+1} grows indefinitely smaller about a point P , by writing

$$S_{r+1} = \int_{S_{r+1}} dS_{r+1}$$

we shall have

$$\lim_{S_{r+1}} \frac{\phi | [S_r]}{S_{r+1}} = \sum_q \Lambda_{q_1 q_2 \dots q_{r+1}} \beta_{q_1 q_2 \dots q_{r+1}} = \frac{d\phi}{dS_{r+1}}$$

where the β are the direction cosines of S_{r+1} at P .

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Let us take $S_{r+1} = S_{r+1}^{(i_1 i_2 \dots i_{r+1})}$ such that at P all the direction cosines β shall be zero except $\beta_{i_1 i_2 \dots i_{r+1}} = 1$. We shall have

$$\lim_{S_{r+1}} \frac{\phi [S_r]}{S_{r+1}^{(i_1 i_2 \dots i_{r+1})}} = \Lambda_{i_1 i_2 \dots i_{r+1}}.$$

Therefore we shall write

$$\Lambda_{i_1 i_2 \dots i_{r+1}} = \frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})},$$

and define this quantity as the *derivative of ϕ with respect to $x_{i_1} x_{i_2} \dots x_{i_{r+1}}$* . What relations must these derivatives satisfy? Before proceeding to the search for these relations, it will be necessary to give an extension of Stokes's theorem, a subject which is dealt with in the next section.

4. Extension of Stokes's theorem

I. Let $L_{i_1 i_2 \dots i_r}$ be functions of the points of the hyperspace S_n , such that every transposition of the indices creates a change of sign, and form the expression

$$(I) \quad M_{i_1 i_2 \dots i_{r+1}} = \sum_s^{r+1} (-1)^{s-1} \frac{\partial L_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}}}{\partial x_{i_s}}.$$

Let S_{r+1} be a hyperspace, of $r+1$ dimensions, bounded by a set of hyperspaces S_r , let its direction cosines be $a_{i_1 i_2 \dots i_{r+1}}$, and form the expression

$$\int_{S_{r+1}} \Omega dS_{r+1},$$

putting $\Omega = \Sigma_i M_{i_1 i_2 \dots i_{r+1}} a_{i_1 i_2 \dots i_{r+1}}$.

If the equations of S_{r+1} are

$$x_i = x_i(\omega_1, \omega_2 \dots \omega_{r+1}) \quad (i = 1, 2, \dots n),$$

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we shall have

$$\begin{aligned}
 \Omega dS_{r+1} &= \sum_i M_{i_1 i_2 \dots i_{r+1}} \frac{d(x_{i_1} x_{i_2} \dots x_{i_{r+1}})}{d(\omega_1 \omega_2 \dots \omega_{r+1})} d\omega_1 d\omega_2 \dots d\omega_{r+1} \\
 &= \sum_i \sum_{i_s}^n \frac{\partial L_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}}}{\partial x_{i_s}} \frac{d(x_{i_s} x_{i_1} \dots x_{i_{s-1}} x_{i_{s+1}} \dots x_{i_{r+1}})}{d(\omega_1 \omega_2 \dots \omega_{r+1})} d\omega_1 d\omega_2 \dots d\omega_{r+1} \\
 &= \sum_i \frac{d(L_{i_1 \dots i_r}, x_{i_1} x_{i_2} \dots x_{i_r})}{d(\omega_1, \omega_2, \dots \omega_{r+1})} d\omega_1 d\omega_2 \dots d\omega_{r+1} \\
 &= \sum_i \sum_{i=1}^{r+1} (-1)^{i-1} \frac{\partial L_{i_1 \dots i_r}}{\partial \omega_i} \frac{d(x_1 \dots x_r)}{d(\omega_1 \dots \omega_{i-1} \omega_{i+1} \dots \omega_{r+1})} d\omega_1 d\omega_2 \dots d\omega_{r+1}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \int_{S_{r+1}} \Omega dS_{r+1} &= \int_{S_r} \sum_i \sum_{i=1}^{r+1} L_{i_1 i_2 \dots i_r} \frac{d(x_{i_1} x_{i_2} \dots x_{i_r})}{d(\omega_i \dots \omega_{i-1} \omega_{i+1} \dots \omega_{r+1})} d\omega_i \dots d\omega_{i-1} d\omega_{i+1} \dots d\omega_{r+1}.
 \end{aligned}$$

We can make the hyperspace S depend on r independent parameters $\bar{\omega}_1, \bar{\omega}_2 \dots \bar{\omega}_r$, whence we shall have

$$\int_{S_{r+1}} \Omega dS_{r+1} = \int_{S_r} \sum_i L_{i_1 i_2 \dots i_r} \frac{d(x_{i_1} x_{i_2} \dots x_{i_r})}{d(\bar{\omega}_1 \bar{\omega}_2 \dots \bar{\omega}_r)} d\bar{\omega}_1 d\bar{\omega}_2 \dots d\bar{\omega}_r.$$

From this comes the formula

$$(2) \quad \int_{S_{r+1}} \sum_i M_{i_1 i_2 \dots i_{r+1}} \alpha_{i_1 i_2 \dots i_{r+1}} dS_{r+1} = \int_{S_r} \sum_i L_{i_1 i_2 \dots i_r} \beta_{i_1 i_2 \dots i_r} dS_r,$$

where the β 's are the direction cosines of the hyperspace S_r .

2. From these formulæ it follows that if

$$\int_{S_r} \sum_i L_{i_1 i_2 \dots i_r} \beta_{i_1 i_2 \dots i_r} dS = 0,$$

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for every closed hyperspace S_r in the region S_n , the necessary and sufficient conditions that must be satisfied are

$$(3) \quad M_{i_1 i_2 \dots i_{r+1}} = \sum_{s=1}^{r+1} (-1)^{s-1} \frac{\partial L_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}}}{\partial x_{i_s}} = 0$$

for every combination of the indices $i_1 i_2 \dots i_{r+1}$.

5. *Conditions which the derivatives of functions of hyperspaces must satisfy. Formulæ for the change of coördinates*

1. Let $\phi | [S_r] |$ be regular, and return to formula (5') of section 3. Since the integral which appears in the right-hand member does not change when S_{r+1} changes, provided the boundary S_r does not change, we must have

$$\int_{S_{r+1}} \sum_a \Lambda_{a_1 a_2 \dots a_{r+1}} \beta_{a_1 a_2 \dots a_{r+1}} dS_{r+1} = 0$$

when the integration is extended over any closed hyperspace S_{r+1} . Hence the necessary and sufficient conditions which the Λ must satisfy in order to be the derivatives of a regular function of hyperspaces S_r (see section 4, article 2) is

$$(D) \quad \sum_{s=1}^{r+2} (-1)^{s-1} \frac{\partial \Lambda_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}}{\partial x_{i_s}} = 0$$

for every possible combination of the indices $i_1 i_2 \dots i_{r+2}$. We can write these equations, making use of the symbols of section 3, article 7, in the form

$$(D') \quad \sum_{s=1}^{r+2} (-1)^{s-1} \frac{\partial}{\partial x_{i_s}} \frac{\partial \phi}{\partial (x_{i_1} \dots x_{i_{s-1}} x_{i_{s+1}} \dots x_{i_{r+2}})} = 0.$$

We shall call these conditions the *conditions of integrability*.

2. Consider now the formulæ for change of variable, transforming the variables $x_1, x_2 \dots x_n$ into $x'_1, x'_2 \dots x'_n$ by

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means of the relations

$$x'_i = x'_i(x_1, x_2, \dots, x_n) \quad (i = 1, 2, \dots, n)$$

such that

$$\frac{d(x'_1, x'_2, \dots, x'_n)}{d(x_1, x_2, \dots, x_n)}$$

is always finite and different from zero. Let us consider two regions which correspond in a one-to-one manner, S_n and S'_n , one belonging to the first set of variables, the other to the second. Let S_{r+1} be a hyperspace, bounded by S_r and contained in S_n , and let S'_{r+1} , bounded by S'_r , correspond to it in S'_n . If we suppose that S_{r+1} is given by the equations

$$x_i = x_i(\omega_1 \omega_2 \dots \omega_{r+1}) \quad (i = 1, 2, \dots, n),$$

we shall have

$$\begin{aligned} \phi | [S_r] | &= \int_{S_{r+1}} \sum_i \frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})} \frac{d(x_{i_1} \dots x_{i_{r+1}})}{d(\omega_1 \dots \omega_{r+1})} d\omega_1 d\omega_2 \dots d\omega_{r+1} \\ &= \int_{S_{r+1}} \sum_i \frac{\partial \phi}{\partial (x_{i_1} \dots x_{i_{r+1}})} \sum_n \frac{d(x_{i_1} \dots x_{i_{r+1}})}{d(x'_{h_1} \dots x'_{h_{r+1}})} \frac{d(x'_{h_1} \dots x'_{h_{r+1}})}{d(\omega_1 \dots \omega_{r+1})} d\omega_1 \dots d\omega_{r+1} \\ &= \int_{S_{r+1}} \sum_h \frac{d(x'_{h_1} \dots x'_{h_{r+1}})}{d(\omega_1 \dots \omega_{r+1})} \sum_i \frac{\partial \phi}{\partial (x_{i_1} \dots x_{i_{r+1}})} \frac{d(x_{i_1} \dots x_{i_{r+1}})}{d(x'_{h_1} \dots x'_{h_{r+1}})} d\omega_1 \dots d\omega_{r+1} \\ &= \int_{S'_{r+1}} \sum_h \beta'_{h_1 \dots h_{r+1}} \left(\sum_i \frac{\partial \phi}{\partial (x_{i_1} \dots x_{i_{r+1}})} \frac{d(x_{i_1} \dots x_{i_{r+1}})}{d(x'_{h_1} \dots x'_{h_{r+1}})} \right) dS'_{r+1} \end{aligned}$$

where the β' denote the direction cosines of S'_{r+1} .

If we write

$$\Lambda'_{h_1 h_2 \dots h_{r+1}} = \sum_i \frac{\partial \phi}{\partial (x_{i_1} \dots x_{i_{r+1}})} \frac{d(x_{i_1} \dots x_{i_{r+1}})}{d(x'_{h_1} \dots x'_{h_{r+1}})}$$

we shall have

$$\phi | [S_r] | = \phi | [S'_r] | = \int_{S'_{r+1}} \sum_h \Lambda'_{h_1 h_2 \dots h_{r+1}} \beta'_{h_1 h_2 \dots h_{r+1}} dS'_{r+1},$$

whence

$$\Lambda'_{h_1 \dots h_{r+1}} = \frac{\partial \phi}{\partial (x'_{h_1} x'_{h_2} \dots x'_{h_{r+1}})}.$$

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The desired formulæ for the transformation of coördinates become then

$$(1) \quad \frac{\partial \phi}{\partial (x'_{h_1} x'_{h_2} \dots x'_{h_{r+1}})} = \sum_i \frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})} \frac{d(x_{i_1} x_{i_2} \dots x_{i_{r+1}})}{d(x'_{h_1} x'_{h_2} \dots x'_{h_{r+1}})}.$$

3. If we multiply the preceding equations by

$$\frac{d(x_{s_{r+2}} x_{s_{r+3}} \dots x_{s_n})}{d(x'_{h_{r+2}} x'_{h_{r+3}} \dots x'_{h_n})}$$

and add them, for all values of the h 's, we shall have

$$\begin{aligned} \sum_h \frac{\partial \phi}{\partial (x'_{h_1} x'_{h_2} \dots x'_{h_{r+1}})} \frac{d(x_{s_{r+2}} x_{s_{r+3}} \dots x_{s_n})}{d(x'_{h_{r+2}} x'_{h_{r+3}} \dots x'_{h_n})} \\ = \frac{\partial \phi}{\partial (x_{s_1} x_{s_2} \dots x_{s_{r+1}})} \frac{d(x_1 x_2 \dots x_n)}{d(x'_1 x'_2 \dots x'_n)} \end{aligned}$$

where

$$(h_1, h_2 \dots h_{r+1}, h_{r+2} \dots h_n) \equiv (s_1, s_2 \dots s_{r+1}, s_{r+2} \dots s_n) \equiv (1, 2, \dots n),$$

the notation being used to denote the fact that the groups of the h 's and of the s 's are two even permutations of the first m integers. Hence

$$\begin{aligned} (2) \quad \frac{\partial \phi}{\partial (x_{s_1} x_{s_2} \dots x_{s_{r+1}})} \\ = \frac{1}{\frac{d(x_1 x_2 \dots x_n)}{d(x'_1 x'_2 \dots x'_n)}} \sum_h \frac{\partial \phi}{\partial (x'_{h_1} x'_{h_2} \dots x'_{h_{r+1}})} \frac{d(x_{s_{r+2}} x_{s_{r+3}} \dots x_{s_n})}{d(x'_{h_{r+2}} x'_{h_{r+3}} \dots x'_{h_n})}. \end{aligned}$$

4. By means of the equations (D'), which are satisfied by the functions $\frac{\partial \phi}{\partial (x_{s_1} \dots x_{s_{r+1}})}$, and the analogous equations satisfied by the functions $\frac{\partial \phi}{\partial (x'_{h_1} \dots x'_{h_{r+1}})}$, we obtain the theorem :

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If the quantities $a_{i_1 i_2 \dots i_{r+1}}$ (which change sign for every transposition in the indices) satisfy the equations

$$(3) \quad \sum_1^{r+2} (-1)^{s-1} \frac{\partial a_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}}{\partial x_{i_s}} = 0$$

the quantities $a'_{h_1 h_2 \dots h_{r+1}}$ given by the formulæ

$$a'_{h_1 h_2 \dots h_{r+1}} = \frac{1}{\frac{d(x'_1 \dots x'_n)}{d(x_1 \dots x_n)}} \sum_i a_{i_1 i_2 \dots i_{r+1}} \frac{d(x'_{h_{r+2}} x'_{h_{r+3}} \dots x'_{h_n})}{d(x_{i_{r+2}} x_{i_{r+3}} \dots x_{i_n})}$$

$$(i_1, i_2 \dots i_n) \equiv (h_1, h_2 \dots h_n) \equiv (1, 2, \dots n)$$

will satisfy the analogous equations

$$(3') \quad \sum_1^{r+2} (-1)^{s-1} \frac{\partial a'_{h_1 \dots h_{s-1} h_{s+1} \dots h_{r+2}}}{\partial x'_{h_s}} = 0.$$

5. Let us write $\frac{\partial \phi}{\partial (x_{i_1} \dots x_{i_{r+1}})} = a_{i_1 \dots i_{r+1}}$.

We wish to show that if the following conditions are satisfied

$$(4) \quad \sum_1^{r+2} (-1)^s a_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} a_{i_s h_1 \dots h_r} = 0$$

and we make a change of variables from the $x_1, x_2 \dots x_n$ to the $x'_1, x'_2, \dots x'_n$, we shall obtain the result that the quantities

$$a'_{h_1 \dots h_{r+1}} = \frac{\partial \phi}{\partial (x'_{h_1} \dots x'_{h_{r+1}})}$$

will satisfy the analogous equations

$$(4') \quad \sum_1^{r+2} (-1)^s a'_{h_1 \dots h_{s-1} h_{s+1} \dots h_{r+2}} a'_{h_s i_1 \dots i_r} = 0.$$

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In fact if we have the relations (4), the $a_{t_1 \dots t_{r+1}}$ will be minor determinants of a matrix

$$\left\| \begin{array}{cccccc} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{r+1,1} & A_{r+1,2} & \cdots & A_{r+1,n} \end{array} \right\|,$$

that is, we can write

$$a_{t_1 \dots t_{r+1}} = \begin{vmatrix} A_{1,t_1} & \cdots & A_{1,t_{r+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{r+1,t_1} & \cdots & A_{r+1,t_{r+1}} \end{vmatrix}.$$

If we write $\frac{\partial x_s}{\partial x'_t} = B_{ts}$

we shall have, by means of (1), the equations

$$a'_{h_1 \dots h_{r+1}} = \sum_{t_i} \begin{vmatrix} A_{1,t_1} & \cdots & A_{1,t_{r+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{r+1,t_1} & \cdots & A_{r+1,t_{r+1}} \end{vmatrix} \begin{vmatrix} B_{h_1 t_1} & \cdots & B_{h_1 t_{r+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{h_{r+1} t_1} & \cdots & B_{h_{r+1} t_{r+1}} \end{vmatrix}$$

that is, if we define $\sum_s A_{ts} B_{hs} = C_{th}$, the relations

$$a'_{h_1 \dots h_{r+1}} = \begin{vmatrix} C_{1,h_1} & \cdots & C_{1,h_{r+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{r+1,h_1} & \cdots & C_{r+1,h_{r+1}} \end{vmatrix}$$

In other words, the quantities $a'_{h_1 \dots h_{r+1}}$ are minor determinants of the matrix

$$\left\| \begin{array}{cccccc} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{r+1,1} & C_{r+1,2} & \cdots & C_{r+1,n} \end{array} \right\|$$

and so the equations (4') will be satisfied.

When the equations (4) are satisfied, the function $\phi | [S_r] |$ is said to be *elementary* (see §§ 10, 14).

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6. Isogeneity *

1. Two complex functions f, ϕ , of hyperspaces S_n , which are *regular*, are said to be *isogenous* if in every point of the total hyperspace S_n , the ratio

$$\frac{\frac{d\phi}{dS_{r+1}}}{\frac{df}{dS_{r+1}}}$$

is independent of the hyperspace S .

Separating the real and imaginary parts, let us write

$$\frac{\partial f}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})} = p_{i_1 \dots i_{r+1}} + i q_{i_1 \dots i_{r+1}} = p_I + i q_I$$

$$\frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})} = \omega_{i_1 \dots i_{r+1}} + i \chi_{i_1 \dots i_{r+1}} = \bar{\omega}_I + i \chi_I$$

where I denotes the set of indices $i_1 i_2 \dots i_{r+1}$, that is, $I \equiv (i_1 \dots i_{r+1})$. The necessary and sufficient condition in order that f and ϕ be isogenous may be written

$$(1) \quad \frac{\bar{\omega}_I + \chi_I}{p_I + q_I} = \frac{\bar{\omega}_H + i \chi_H}{p_H + i q_H},$$

where $H \equiv (h_1 h_2 \dots h_{r+1})$ is another arbitrary combination of the indices. From the preceding equations we find

$$(2) \quad \begin{cases} \bar{\omega}_I p_H - \bar{\omega}_H p_I = \chi_I q_H - \chi_H q_I, \\ \bar{\omega}_I q_H - \bar{\omega}_H q_I = \chi_H p_I - \chi_I p_H. \end{cases}$$

$$2. \text{ Let us write } p_I p_H + q_I q_H = E_{I, H},$$

$$p_I q_H - p_H q_I = D_{I, H}.$$

* See my note: "Sopra una estensione della teoria di Riemann sulle funzioni di variabile complessa." ("Atti della R. Acc. dei Lincei," Vol. III, fasc. 10.)

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Among the E 's and D 's we shall have the relations

$$(3) \quad D_{IH}E_{LK} + D_{HK}E_{LI} + D_{KI}E_{LH} = 0,$$

$$\begin{vmatrix} E_{IH}E_{IL} \\ E_{KH}E_{KL} \end{vmatrix} = \begin{vmatrix} p_I p_H + q_I q_H & p_I p_L + q_I q_L \\ p_K p_H + q_K q_H & p_K p_L + q_K q_L \end{vmatrix} = \begin{vmatrix} p_I q_I & p_H q_H \\ p_K q_K & p_L q_L \end{vmatrix} = D_{IK}D_{HL}$$

whence

$$(4) \quad E_{IH}E_{KL} - E_{KH}E_{IL} = D_{IK}D_{HL}.$$

3. If we solve the equations (2) for $\bar{\omega}_I$ and χ_I , we shall have

$$\bar{\omega}_I = \frac{E_{IH}\chi_I - E_{II}\chi_H}{D_{HI}}, \quad \chi_I = \frac{E_{IH}\bar{\omega}_I - E_{II}\bar{\omega}_H}{D_{IH}}$$

Since, however, the first member of these equations does not depend on H , we must have

$$\begin{aligned} \bar{\omega}_I &= \frac{E_{IH}\chi_I - E_{II}\chi_H}{D_{HI}} = \frac{E_{IK}\chi_I - E_{II}\chi_K}{D_{KI}} \\ &= \frac{(E_{IH}\chi_I - E_{II}\chi_H)E_{IK} - (E_{IK}\chi_I - E_{II}\chi_K)E_{IH}}{D_{HI}E_{IK} - D_{KI}E_{IH}} \\ &= \frac{E_{IH}\chi_K - E_{IK}\chi_H}{D_{HK}}. \end{aligned}$$

In a similar way we can operate on the expression for χ_I , and therefore whatever H and K may be we have the formulæ

$$(E) \quad \bar{\omega}_I = \frac{E_{IH}\chi_K - E_{IK}\chi_H}{D_{HK}}, \quad \chi_I = \frac{E_{IH}\bar{\omega}_K - E_{IK}\bar{\omega}_H}{D_{KH}}$$

4. From the preceding formulæ it follows that

$$D_{HK}\bar{\omega}_I = E_{IH}\chi_K - E_{IK}\chi_H,$$

$$D_{KI}\bar{\omega}_H = E_{HK}\chi_I - E_{HI}\chi_K,$$

$$D_{IH}\bar{\omega}_K = E_{KI}\chi_H - E_{KH}\chi_I,$$

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hence, whatever I, H, K may be, we have the formula

$$(F) \quad D_{HK}\bar{\omega}_I + D_{KI}\bar{\omega}_H + D_{IH}\bar{\omega}_K = 0,$$

and similarly, $D_{HK}\chi_I + D_{KI}\chi_H + D_{IH}\chi_K = 0$.

5. Let us return to the equations (E); from them it follows that

$$(5) \quad \Theta_{IL} = \frac{I}{D_{IL}} \begin{vmatrix} \bar{\omega}_I & \chi_I \\ \bar{\omega}_L & \chi_L \end{vmatrix} = \frac{\chi_K E_{IH} - \chi_H E_{IK}}{D_{IL} D_{HK}} \chi_L - \frac{\chi_K E_{LH} - \chi_H E_{LK}}{D_{IL} D_{HK}} \chi_I \\ = \frac{E_{IH}\chi_K\chi_L - E_{IK}\chi_H\chi_L + E_{LK}\chi_H\chi_I - E_{LH}\chi_K\chi_I}{D_{IL} D_{HK}}.$$

If we interchange I with H and L with K the last member of this equation will not change. Hence we shall have

$$(6) \quad \frac{I}{D_{IL}} \begin{vmatrix} \bar{\omega}_I & \chi_I \\ \bar{\omega}_L & \chi_L \end{vmatrix} = \frac{I}{D_{HK}} \begin{vmatrix} \bar{\omega}_H & \chi_H \\ \bar{\omega}_K & \chi_K \end{vmatrix}.$$

In other words, the quantities Θ_{IL} are independent of I and L , and so we can denote them all by Θ .

If in (5) we put $I = H, L = K$, we shall have

$$(7) \quad \Theta = \frac{E_{HH}\chi_L^2 - 2 E_{HL}\chi_I\chi_L + E_{LL}\chi_I^2}{D_{HL}^2} \\ = \frac{(p_I\chi_L - p_L\chi_I)^2 + (q_I\chi_L - q_L\chi_I)^2}{D_{HL}^2}$$

formulæ which show that Θ is a *positive* quantity. If in (5) we interchange $\bar{\omega}$ and χ , and p and q , the Θ will not change, and we shall have for Θ the alternative expression

$$(5') \quad \Theta = \frac{E_{IH}\bar{\omega}_K\bar{\omega}_L - E_{IK}\bar{\omega}_H\bar{\omega}_L + E_{LK}\bar{\omega}_H\bar{\omega}_I - E_{LH}\bar{\omega}_K\bar{\omega}_I}{D_{IL} D_{HK}}.$$

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If we write $\phi = \phi_1 + i\phi_2$ and make use of our symbols $I, H \dots$, we can write

$$\omega_I = \bar{\omega}_{i_1 \dots i_{r+1}} = \frac{\partial \phi_1}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})} = \frac{\partial \phi_1}{\partial (x_I)},$$

$$\chi_I = \chi_{i_1 \dots i_{r+1}} = \frac{\partial \phi_2}{\partial (x_{i_1} x_{i_2} \dots x_{i_{r+1}})} = \frac{\partial \phi_2}{\partial (x_I)}$$

where (x_I) is a substitute for $(x_{i_1} x_{i_2} \dots x_{i_{r+1}})$, *i.e.*

$$(x_I) \equiv (x_{i_1} x_{i_2} \dots x_{i_{r+1}}).$$

The expression for Θ can now be written

$$(G) \quad \Theta =$$

$$\frac{E_{IH} \frac{\partial \psi}{\partial (x_I)} \frac{\partial \psi}{\partial (x_L)} - E_{IK} \frac{\partial \psi}{\partial (x_H)} \frac{\partial \psi}{\partial (x_K)} + E_{LK} \frac{\partial \psi}{\partial (x_H)} \frac{\partial \psi}{\partial (x_I)} - E_{LH} \frac{\partial \psi}{\partial (x_K)} \frac{\partial \psi}{\partial (x_I)}}{D_{IL} D_{HK}}$$

where in place of ψ we can put either ϕ_1 or ϕ_2 .

6. We know that the quantities $\bar{\omega}$ and χ must satisfy the following equations (see section 5, article 1)

$$\sum_1^{r+2} (-1)^{s-1} \frac{\partial}{\partial x_s} \bar{\omega}_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} = 0, \quad \sum_1^{r+2} (-1)^{s-1} \frac{\partial}{\partial x_s} \chi_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}},$$

and therefore, from (E), we have the following equations

$$(H) \quad \left\{ \begin{array}{l} \sum_1^{r+2} (-1)^{s-1} \frac{\partial}{\partial x_s} \\ \left\{ \frac{\chi_K E_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}^H - \chi_H E_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}^K}{D_{HK}} \right\} = 0 \\ \sum_1^{r+2} (-1)^{s-1} \frac{\partial}{\partial x_s} \\ \left\{ \frac{\bar{\omega}_R E_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}^H - \bar{\omega}_H E_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}^K}{D_{HK}} \right\} = 0 \end{array} \right.$$

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or, by reason of (H) and (F), ϕ_1 and ϕ_2 must satisfy the equations

$$(H') \quad \sum_1^{r+2} (-1)^{s-1} \frac{\partial}{\partial x_s} \left[\frac{E_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}, H} \frac{\partial \psi}{\partial (x_K)} - E_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}, K} \frac{\partial \psi}{\partial (x_H)}}{D_{HK}} \right] = 0$$

$$(F') \quad D_{HK} \frac{\partial \psi}{\partial (x_I)} + D_{KI} \frac{\partial \psi}{\partial (x_H)} + D_{IH} \frac{\partial \psi}{\partial (x_K)} = 0.$$

7. Conversely it can be shown that if $\psi|[S_r]|$ is a real regular function and satisfies the preceding equations, it may be considered as the real part of a function $\psi + i\theta$ isogenous to f . In fact, by means of (H') we can write

$$\frac{E_{I, H} \frac{\partial \psi}{\partial (x_K)} - E_{I, K} \frac{\partial \psi}{\partial (x_H)}}{D_{HK}} = \frac{\partial \theta_{H, K}}{\partial (x_I)}$$

where $(x_I) = (x_{i_1} \dots x_{i_{s-1}} x_{i_{s+1}} \dots x_{i_{r+2}})$. But from (F') and (3) it follows that the first member of the preceding equations is independent of H and K , hence we can take the θ_{HK} as independent of their subscripts and write them all equal to θ , so that

$$\frac{E_{IH} \frac{\partial \psi}{\partial (x_K)} - E_{IK} \frac{\partial \psi}{\partial (x_H)}}{D_{HK}} = \frac{\partial \theta}{\partial (x_I)}.$$

And now if from these equations we follow the inverse procedure to that of articles 1, 2, 3, we find that the ratio

$$\frac{\frac{\partial (\psi + i\theta)}{\partial (x_I)}}{p_I + iq_I}$$

is independent of the indices (I), so that $\psi + i\theta$ is isogenous

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to f . The equations (H') and (F') operate in our case in the same way as the equation $\Delta^2 = 0$ in the theory of Riemann.

7. Conditions for isogeneity.

1. If we take arbitrarily a regular function of hyper-spaces S_r , it will not always be possible to associate with it an isogenous function. In order for that it is necessary that certain conditions be satisfied. In fact if $F|[S_r]|$ is a regular function to which $\Phi|[S_r]|$ is isogenous, and we write

$$\frac{\partial F}{\partial(x_{i_1} \dots x_{i_{r+1}})} = p_{i_1 \dots i_{r+1}}, \quad \frac{\partial \Phi}{\partial(x_{i_1} \dots x_{i_{r+1}})} = \bar{\omega}_{i_1 \dots i_{r+1}},$$

we must have

$$\frac{\bar{\omega}_{i_1 \dots i_{r+1}}}{p_{i_1 \dots i_{r+1}}} = \phi$$

where ϕ is independent of the indices $i_1 \dots i_{r+1}$. Hence it follows that

$$\bar{\omega}_{i_1 \dots i_{r+1}} = \phi p_{i_1 \dots i_{r+1}}$$

so that

$$\begin{aligned} 0 &= \sum_1^{r+2} (-1)^s \frac{\partial \bar{\omega}_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}}{\partial x_{i_s}} = \sum_1^{r+2} (-1)^s \frac{\partial (\phi p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}})}{\partial x_{i_s}} \\ &= \sum_1^{r+2} (-1)^s p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} \frac{\partial \phi}{\partial x_{i_s}}. \end{aligned}$$

From this we conclude that *it is necessary and sufficient in order that there may exist a function isogenous to $F|[S_r]|$ that the system of simultaneous linear differential equations*

$$(I) \quad \sum_1^{r+2} (-1)^s p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} \frac{\partial \phi}{\partial x_{i_s}} = 0$$

admit solutions.

It is for this reason that in § 9 we shall study systems of differential equations of this form. In the meantime let us observe that the equations (I) may in some cases be incompatible. Thus, if we have in four dimensions the regular function $F|[S_1]|$, the equations (I) become

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$$-p_{23} \frac{\partial \phi}{\partial x_1} + p_{13} \frac{\partial \phi}{\partial x_2} - p_{12} \frac{\partial \phi}{\partial x_3} = 0,$$

$$-p_{34} \frac{\partial \phi}{\partial x_2} + p_{24} \frac{\partial \phi}{\partial x_3} - p_{23} \frac{\partial \phi}{\partial x_4} = 0,$$

$$-p_{41} \frac{\partial \phi}{\partial x_3} + p_{31} \frac{\partial \phi}{\partial x_4} - p_{34} \frac{\partial \phi}{\partial x_1} = 0,$$

$$-p_{12} \frac{\partial \phi}{\partial x_4} + p_{42} \frac{\partial \phi}{\partial x_1} - p_{41} \frac{\partial \phi}{\partial x_2} = 0,$$

and these equations will be incompatible unless

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$$

2. We now proceed to prove the following theorem :

The necessary and sufficient condition in order that equations

(1) *admit a common solution* ϕ *is that we can write*

$$(2) \quad p_{i_1 \dots i_{r+2}} = \sum_{s=1}^{r+1} (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \psi}{\partial (x_{i_1} \dots x_{i_{s-1}} x_{i_{s+1}} \dots x_{i_{r+1}})}$$

where ψ is a regular function of hyperspaces.

Let us write
$$\frac{\partial \psi}{\partial (x_{i_1} \dots x_{i_r})} = q_{i_1 \dots i_r},$$

It is easy to show that if the equations

$$(2') \quad p_{i_1 \dots i_{r+1}} = \sum_{s=1}^{r+1} (-1)^s \frac{\partial \phi}{\partial x_{i_s}} q_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}}$$

are satisfied, the equations (1) will also be satisfied. In fact, we shall have

$$\begin{aligned} & \sum_{t=1}^{r+2} (-1)^t p_{i_1 \dots i_{t-1} i_{t+1} \dots i_{r+2}} \frac{\partial \phi}{\partial x_{i_t}} \\ &= \sum_{t=1}^{r+2} \sum_{s=1}^{r+2} \binom{r+2}{s} (-1)^{s+t} \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \phi}{\partial x_{i_t}} q_{i_1 \dots i_{s-1} i_{s+1} \dots i_{t-1} i_{t+1} \dots i_{r+1}}, \end{aligned}$$

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in which $\sum_1^{r+2} {}^{(t)}$ is extended over all the values of the index s from 1 to $r+2$, the value t excepted, and s' should be taken equal to s or to $s-1$ according as $s < t$ or $s > t$. Hence the left-hand member of the equation is zero, and the equations (1) are satisfied. From (2') it also follows easily that

$$\sum_1^{r+2} {}^s (-1)^s \frac{\partial p_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}}}{\partial x_{t_s}} = 0.$$

Thus we have shown that our condition is sufficient. To show that it is also necessary, let us execute a change of variables, instead of $x_1, x_2 \dots x_n$ taking $x'_1 = \phi, x'_2 = x_2 \dots x'_n = x_n$. If we prime the letters which refer to the new variables, we shall have

1st) if $i_1, i_2, \dots, i_r \neq 1$

$$q_{t_1 \dots t_r} = q'_{t_1 \dots t_r} + \sum_1^r {}^t (-1)^{t-1} q'_{1 t_1 \dots t_{t-1} t_{t+1} \dots t_r} \frac{\partial \phi}{\partial x_{t_t}}$$

2d) if $i_h = 1$

$$q_{t_1 \dots t_r} = \sum_1^r {}^s (-1)^{s-1} q'_{1 t_1 \dots t_{s-1} t_{s+1} \dots t_r} \frac{\partial \phi}{\partial x_{t_s}} = (-1)^{h-1} q'_{1 t_1 \dots t_{h-1} t_{h+1} \dots t_r} \frac{\partial \phi}{\partial x_{t_h}}.$$

Supposing momentarily that $i_1 \dots i_{r+1} \neq 1$ we shall have

$$\begin{aligned} p_{t_1 \dots t_{r+1}} &= \sum_1^{r+1} {}^s (-1)^s \frac{\partial \phi}{\partial x_{t_s}} q'_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+1}} \\ &+ \sum_1^{r+1} {}^s (-1)^s \frac{\partial \phi}{\partial x_{t_s}} \sum_1^{r+1} {}^{(s)} {}^t (-1)^{t'} q'_{1 t_1 \dots t_{t-1} t_{t+1} \dots t_{s-1} t_{s+1} \dots t_{r+1}} \frac{\partial \phi}{\partial x_{t_t}} \end{aligned}$$

where $t' = \begin{cases} t-1 \\ t \end{cases}$ according as $\begin{cases} t < s \\ t > s. \end{cases}$

Hence

$$(3) \quad p_{t_1 \dots t_{r+1}} = \sum_1^{r+1} {}^s (-1)^s q'_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+1}} \frac{\partial \phi}{\partial x_{t_s}}.$$

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If we suppose instead that some one of the indices of p is equal to 1, say $i_1 = 1$, we shall have

$$(3') \quad p_{1i_2 \dots i_{r+1}} = -\frac{\partial \phi}{\partial x_1} q'_{i_2 \dots i_{r+1}} - \frac{\partial \phi}{\partial x_1} \sum_{s=1}^{r+1} (-1)^s q'_{1i_2 \dots i_{s-1} i_{s+1} \dots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}} \\ + \sum_{s=1}^{r+1} (-1)^s q'_{1i_2 \dots i_{s-1} i_{s+1} \dots i_{r+1}} \frac{\partial \phi}{\partial x_1} \frac{\partial \phi}{\partial x_{i_s}} = -\frac{\partial \phi}{\partial x_1} q'_{i_2 \dots i_{r+1}}.$$

We shall show that (3) is a consequence of (3'). In fact, from (3') we have

$$q'_{i_2 \dots i_{r+1}} = -\frac{p_{1i_2 \dots i_{r+1}}}{\left(\frac{\partial \phi}{\partial x_1}\right)}$$

so that (3) becomes

$$p_{i_1 \dots i_{r+1}} = -\sum_{s=1}^{r+1} (-1)^s \frac{p_{1i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}}}{\frac{\partial \phi}{\partial x_1}} \frac{\partial \phi}{\partial x_{i_s}}$$

and if we put $i_0 = 1$, this gives us

$$\sum_{s=0}^{r+1} (-1)^s p_{i_0 i_1 \dots i_{s-1} i_{s+1} \dots i_r} \frac{\partial \phi}{\partial x_{i_s}} = 0,$$

an equation which is identically true.

We must now prove that the functions

$$q'_{i_2 \dots i_{r+1}} = -\frac{p_{1i_2 \dots i_{r+1}}}{\frac{\partial \phi}{\partial x_1}}$$

satisfy the conditions of integrability (see section 5, article 1), assuming therein that ϕ is constant.

We have in fact (see section 5, article 3)

$$p'_{1i_2 \dots i_{r+1}} = \frac{1}{\frac{d(\phi x_2 \dots x_n)}{d(x_1 x_2 \dots x_n)}} \sum_h p_{h_1 h_2 \dots h_{r+1}} \frac{d(x_{i_{r+2}} \dots x_{i_n})}{d(x_{h_{r+2}} \dots x_{h_n})}$$

where

$$(h_1 h_2 \dots h_{r+1} h_{r+2} \dots h_n) \equiv (1 i_2 \dots i_{r+1} i_{r+2} \dots i_n) \equiv (1, 2, \dots, n)$$

so that

$$p'_{1i_2 \dots i_{r+1}} = \frac{p_{1i_2 \dots i_{r+1}}}{\left(\frac{\partial \phi}{\partial x_1}\right)}.$$

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If $i_1, i_2 \dots i_{r+1} \neq 1$, we have (see section 5, article 3)

$$\begin{aligned} p'_{i_1 \dots i_{r+1}} &= \frac{1}{\frac{d(\phi x_2 \dots x_n)}{d(x_1 x_2 \dots x_n)}} \sum_h p_{h_1 \dots h_{r+1}} \frac{d(x_{i_{r+2}} \dots x_{i_n})}{d(x_{h_{r+2}} \dots x_{h_n})} \\ &= \frac{1}{\frac{d\phi}{dx_1}} \sum_0^{r+1} (-1)^s p_{i_0 i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}} = 0. \end{aligned}$$

And so if we apply the theorem of section 5, article 3, we shall have

$$\begin{aligned} 0 &= \sum_2^{r+1} (-1)^s \frac{\partial}{\partial x_{i_s}} \left[\frac{p_{i_1, i_2 \dots i_{s-1} i_{s+1} \dots i_{r+1}}}{\left(\frac{\partial \phi}{\partial x_1} \right)} \right] \\ &= \sum_2^{r+1} (-1)^s \frac{\partial}{\partial x_{i_s}} q'_{i_2 \dots i_{s-1} i_{s+1} \dots i_{r+1}}. \end{aligned}$$

The functions q' then satisfy the conditions of integrability, and it will be possible to determine a function ψ which satisfies equations (2). Thus it is shown that the given condition is necessary.

3. Given the F for which (1) is satisfied, the ψ which satisfies (2) is not determined. We shall see how all the ψ 's which satisfy (2) may be found when one of them, ψ_1 , is known. If ψ_1 and ψ satisfy (2), and we write

$$\psi - \psi_1 = \psi_2, \quad \frac{\partial \psi_2}{\partial (x_{i_1} \dots x_{i_r})} = q_{i_1 \dots i_r}^{(2)},$$

we shall have

$$0 = \sum_1^{r+1} (-1)^s \frac{\partial \phi}{\partial x_{i_s}} q_{i_1 \dots i_r}^{(2)}$$

and therefore

$$q_{i_1 \dots i_r}^{(2)} = \sum_1^r (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \Theta}{\partial (x_{i_1} \dots x_{i_{s-1}} x_{i_{s+1}} \dots x_{i_r})},$$

in which Θ is arbitrary.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

Second Lecture

EXPRESSIONS FOR ISOGENOUS FUNCTIONS — AUXILIARY REMARKS ON SYSTEMS OF SIMULTANEOUS DIFFERENTIAL EQUATIONS — ON THE ELEMENTARY FUNCTIONS — COMPOSITION OF FUNCTIONS OF HYPERSPACES — NEW CONSIDERATIONS WITH REFERENCE TO THE RELATION OF ISOGENEITY — DIFFERENTIATION AND INTEGRATION — ISOGENEITY OF ORDER r .

8. Expressions for isogenous functions

1. If $F|[S_r]|$ and $\Phi|[S_r]|$ are isogenous, it follows from what has been shown in the preceding section that we can write

$$\frac{\partial F}{\partial(x_{i_1} \cdots x_{i_{r+1}})} = p_{i_1 \cdots i_{r+1}} = \sum_1^{r+1} (-1)^s \frac{\partial f}{\partial x_{i_s}} \frac{\partial \psi}{\partial(x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_{r+1}})}$$

$$\frac{\partial \Phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} = \bar{\omega}_{i_1 \cdots i_{r+1}} = \sum_1^{r+1} (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \psi}{\partial(x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_{r+1}})},$$

where $\psi|[S_{r-1}]|$ is regular and ϕ is a function of f ; and we know that the ratio $\frac{\bar{\omega}_{i_1 \cdots i_{r+1}}}{p_{i_1 \cdots i_{r+1}}}$ (independent of the indices) is equal to $\frac{d\phi}{df}$.

2. Let us write

$$L_{i_1 \cdots i_{r+1}} = f \frac{\partial \psi}{\partial(x_{i_1} \cdots x_{i_r})}.$$

It follows that

$$(I) \quad p_{i_1 \cdots i_{r+1}} = \sum_1^{r+1} (-1)^s \frac{\partial L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\partial x_{i_s}}.$$

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If now S_{r+1} is a space of $r+1$ dimensions whose boundary is S_r , we shall have

$$F|[S_r]| = \int_{S_{r+1}} \sum_i p_{i_1 \dots i_{r+1}} \alpha_{i_1 \dots i_{r+1}} dS_{r+1},$$

where the $\alpha_{i_1 \dots i_{r+1}}$ are the direction cosines of S_{r+1} . And if we substitute for the p 's their values (1) and apply the extension of Stokes's theorem (see Section 4), we shall have

$$(2) \quad F|[S_r]| = \int_{S_r} f \frac{d\psi}{dS_r} dS_r,$$

and similarly,

$$(2') \quad \Phi|[S_r]| = \int_{S_r} \phi \frac{d\psi}{dS_r} dS_r.$$

3. Conversely, if F and Φ are given by the preceding formulae, with $\phi = \phi(f)$, the F and Φ must be isogenous.

9. Auxiliary remarks on systems of simultaneous differential equations

1. Consider the system of differential equations

$$(I) \quad H_{i_1 i_2 \dots i_{r+2}} = \sum_s^{r+2} (-1)^s A_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} \frac{\partial \phi}{\partial x_{i_s}} = 0$$

whose coefficients satisfy the conditions

$$(2) \quad \sum_s^{r+2} (-1)^s A_{i_s i_1 \dots i_{r+1}} A_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} = 0$$

and are such that they change sign with every transposition of the indices. With this convention, if we have an H with two of its indices equal, its value must be zero.

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2. Among the A 's, one at least must be different from zero. If $A_{i_1 i_2 \dots i_{r+1}}$ is such a one, *all the equations (1) will follow from the equations (independent among themselves).*

$$(3) \quad H_{i_1 \dots i_{r+1} h_1} = 0, \quad H_{i_1 \dots i_{r+1} h_2} = 0, \quad \dots \quad H_{i_1 \dots i_{r+1} h_{n-r-1}} = 0,$$

in which none of the $h_1, h_2 \dots h_{n-r-1}$ is equal to another, or to an i .

Let us take, in fact, the system

$$(4) \quad H_{i_1 \dots i_{r+1} k_1} = 0, \quad H_{i_1 \dots i_{r+1} k_2} = 0, \quad \dots \quad H_{i_1 \dots i_{r+1} k_{r+2}} = 0,$$

where the k_s are arbitrary. If a k_s is equal to one of the i_i , the corresponding equation will be an identity; otherwise, it will be one of the equations (3). The equations (4) can be written in the form

$$A_{i_1 \dots i_{r+1}} \frac{\partial \phi}{\partial x_{k_s}} + \sum_{l=1}^{r+1} (-1)^l A_{k_s i_1 \dots i_{l-1} i_{l+1} \dots i_{r+1}} \frac{\partial \phi}{\partial x_{i_l}} = 0.$$

If we multiply each one by $(-1)^s A_{k_1 \dots k_{s-1} k_{s+1} \dots k_{r+2}}$ and add them together for all values of the subscript s from 1 to $r+2$, we shall have

$$\begin{aligned} & A_{i_1 \dots i_{r+1}} \sum_{s=1}^{r+2} (-1)^s A_{k_1 k_2 \dots k_{s-1} k_{s+1} \dots k_{r+2}} \frac{\partial \phi}{\partial x_{k_s}} \\ & + \sum_{l=1}^{r+1} (-1)^l \frac{\partial \phi}{\partial x_{i_l}} \sum_{s=1}^{r+2} A_{k_s i_1 \dots i_{l-1} i_{l+1} \dots i_{r+2}} A_{k_1 \dots k_{s-1} k_{s+1} \dots k_{r+2}} = 0, \end{aligned}$$

$$\text{whence } \sum_{s=1}^{r+2} (-1)^s A_{k_1 k_2 \dots k_{s-1} k_{s+1} \dots k_{r+2}} \frac{\partial \phi}{\partial x_{k_s}} = H_{k_1 \dots k_{r+2}} = 0$$

so that the theorem is proved.

3. Now let us form the alternating function of Poisson

$$(H_{i_1 i_2 \dots i_{r+2}}, H_{h_1 h_2 \dots h_{r+2}})$$

taking $i_1 = h_1, i_2 = h_2, \dots, i_{r+1} = h_{r+1}$ and writing $h_{r+2} = i_{r+3}$,

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we shall have

$$\begin{aligned}
 & (H_{h_1 \dots h_{r+2}}, H_{t_1 \dots t_{r+2}}) \\
 &= \sum_1^{r+2} \sum_s^{r+2} (-1)^{s+t} \left(A_{h_1 \dots h_{s-1} h_{s+1} \dots h_{r+2}} \frac{\partial A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+2}}}{\partial x_{h_s}} \frac{\partial \phi}{\partial x_{t_t}} \right) \\
 &- \sum_1^{r+2} (-1)^{s+t} \left(A_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}} \frac{\partial A_{h_1 \dots h_{t-1} h_{t+1} \dots h_{r+2}}}{\partial x_{t_s}} \frac{\partial \phi}{\partial x_{h_t}} \right) \\
 &= \sum_1^{r+1} \sum_s^{r+1} (-1)^{s+t} \frac{\partial (A_{t_1 \dots t_{r+1}} A_{t_1 \dots t_{s-1} t_{s+1} \dots t_{t-1} t_{t+1} \dots t_{r+3}})}{\partial x_{t_s}} \frac{\partial \phi}{\partial x_{t_t}} \\
 &- \sum_1^{r+2} (-1)^s \frac{\partial A_{h_1 \dots h_{s-1} h_{s+1} \dots h_{r+2}}}{\partial x_{t_s}} \sum_1^{r+2} (-1)^t A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+3}} \frac{\partial \phi}{\partial x_{t_t}} \\
 &+ \sum_1^{r+2} (-1)^s \frac{\partial A_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}}}{\partial x_{t_s}} \sum_1^{r+2} (-1)^t A_{h_1 \dots h_{t-1} h_{t+1} \dots h_{r+2}} \frac{\partial \phi}{\partial x_{h_t}} \\
 &+ \sum_1^{r+2} (-1)^{t+r+2} \frac{\partial (A_{t_1 \dots t_{r+1}} A_{t_1 \dots t_{t-1} t_{t+1} \dots t_{r+2}})}{\partial x_{t_{r+2}}} \frac{\partial \phi}{\partial x_{t_t}} \\
 &- \sum_1^{r+2} (-1)^{t+r+2} \frac{\partial (A_{t_1 \dots t_{r+1}} A_{h_1 \dots h_{t-1} h_{t+1} \dots h_{r+2}})}{\partial x_{t_{r+2}}} \frac{\partial \phi}{\partial x_{h_t}} \\
 &+ \sum_1^{r+2} (-1)^{s+r+2} \frac{\partial (A_{t_1 \dots t_{r+1}} A_{h_1 \dots h_{s-1} h_{s+1} \dots h_{r+2}})}{\partial x_{h_s}} \frac{\partial \phi}{\partial x_{t_{r+2}}} \\
 &- \sum_1^{r+2} (-1)^{s+r+2} \frac{\partial (A_{t_1 \dots t_{r+1}} A_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}})}{\partial x_{t_s}} \frac{\partial \phi}{\partial x_{h_{r+2}}}.
 \end{aligned}$$

If we write

$$\sum_1^{r+2} (-1)^s \frac{\partial A_{h_1 \dots h_{s-1} h_{s+1} \dots h_{r+2}}}{\partial x_{h_s}} = L_{h_1 \dots h_{r+2}},$$

we shall have

$$\begin{aligned}
 & (H_{h_1 \dots h_{r+2}}, H_{t_1 \dots t_{r+2}}) \\
 &= A_{t_1 \dots t_{r+1}} \sum_1^{r+3} (-1)^s L_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+3}} \frac{\partial \phi}{\partial x_{t_s}} \\
 &+ \sum_1^{r+3} (-1)^s \frac{\partial A_{t_1 \dots t_{r+1}}}{\partial x_{t_s}} H_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+3}} + L_{t_1 \dots t_{r+2}} H_{t_1 \dots t_{r+1} t_{r+3}} \\
 &\quad - L_{t_1 \dots t_{r+1} t_{r+3}} H_{t_1 \dots t_{r+2}}.
 \end{aligned}$$

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Hence to the system (I) we must add the equations

$$\sum_{s=1}^{r+1} (-1)^s L_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+3}} \frac{\partial \phi}{\partial x_{i_s}} = 0$$

so that if the conditions

$$L_{i_1 \dots i_{r+2}} = \sum_{s=1}^{r+2} (-1)^s \frac{\partial A_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}}{\partial x_{i_s}} = 0$$

are satisfied for every combination of the indices $i_1 \dots i_{r+2}$, the system (I) will be complete.

4. From this it follows that the equations (I) of section 7 will form a complete system whenever, in addition to the conditions of integrability (see section 5, article 1), the functions p satisfy also the following conditions:

$$(5) \quad \sum_{s=1}^{r+2} (-1)^s p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} p_{i_s i_1 \dots i_r} = 0.$$

Hence for *elementary* functions (see section 5, article 5) the system of equations (I) of section 7 is *complete*.

10. The elementary functions

1. Let us suppose that the function $F | [S_r] |$ is regular and elementary, so that the system (I) of section 7, or the equivalent system (3) of section 9, is complete. There will exist then $r+1$ independent integrals

$$\phi, \phi_1, \dots, \phi_r.$$

Hence the ratio

$$\theta = \frac{p_{i_1 \dots i_r}}{\left(\frac{d(\phi, \phi_1, \dots, \phi_r)}{d(x_{i_1} \dots x_{i_{r+1}})} \right)} = \frac{\left(\frac{dF}{d(x_{i_1} \dots x_{i_{r+1}})} \right)}{\left(\frac{d(\phi, \phi_1 \dots \phi_r)}{d(x_{i_1} \dots x_{i_{r+1}})} \right)}$$

will be independent of the subscripts $i_1 \dots i_r$, and we shall have

$$p_{i_1 \dots i_{r+1}} = \theta \frac{d(\phi, \phi_1 \dots \phi_r)}{d(x_{i_1} \dots x_{i_{r+1}})}.$$

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But we must have $\sum_1^{r+2} (-1)^s \frac{\partial p_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}}}{\partial x_{t_s}} = 0$,

so that $\sum_1^{r+2} (-1)^s \frac{\partial \theta}{\partial x_{t_s}} \frac{d(\phi, \phi_1 \dots \phi_r)}{d(x_{t_1} \dots x_{t_{s-1}} x_{t_{s+1}} \dots x_{t_{r+2}})} = 0$,

and consequently $\sum_1^{r+2} (-1)^s p_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}} \frac{\partial \theta}{\partial x_{t_s}} = 0$.

The quantity θ will therefore be a function of $\phi_0, \phi_1, \dots, \phi_r$,

and if we write $\frac{\partial \phi_0}{\partial \phi} = \theta$, we shall have

$$p_{t_1 \dots t_{r+1}} = \frac{\partial \phi_0}{\partial \phi} \frac{d(\phi, \phi_1 \dots \phi_r)}{d(x_{t_1} \dots x_{t_{r+1}})} = \frac{d(\phi_0, \phi_1 \dots \phi_r)}{d(x_{t_1} \dots x_{t_{r+1}})}.$$

We have therefore the following theorem :

If F is an elementary function, it follows that

$$\frac{\partial F}{\partial(x_{t_1} \dots x_{t_{r+1}})} = \frac{d(\phi_0, \phi_1 \dots \phi_r)}{d(x_{t_1} \dots x_{t_{r+1}})} = p_{t_1 \dots t_{r+1}}$$

where $\phi_0, \phi_1, \dots, \phi_r$ are independent integrals of the complete system

$$(I) \quad \sum_1^{r+2} (-1)^s p_{t_1 \dots t_{s-1} t_{s+1} \dots t_{r+2}} \frac{\partial \phi}{\partial x_{t_s}} = 0.$$

2. Conversely, if we take $r+1$ functions $\phi_0, \phi_1, \dots, \phi_r$ and write

$$\frac{d(\phi_0, \phi_1, \dots, \phi_r)}{d(x_{t_1} \dots x_{t_{r+1}})} = p_{t_1 \dots t_{r+1}},$$

the quantities $p_{t_1 \dots t_{r+1}}$ will be the derivatives of an elementary function. In fact, they will satisfy the conditions of integrability, and also the conditions (5) of the preceding section (see section 5, article 5).

We shall say that the functions $\phi_0, \phi_1, \dots, \phi_r$ are *conjugate* to the function F , and that F is conjugate to them.

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3. If Φ is isogenous to F , and we write

$$\frac{\partial \Phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} = \bar{\omega}_{i_1 \cdots i_{r+1}},$$

we must have $\frac{\bar{\omega}_{i_1 \cdots i_{r+1}}}{p_{i_1 \cdots i_{r+1}}} = \psi,$

ψ being an integral of equation (1). Hence ψ must be a function of $\phi_0, \phi_1, \cdots \phi_r$. If we take $\psi = \frac{\partial \lambda}{\partial \phi}$, we shall have

$$\bar{\omega}_{i_1 \cdots i_{r+1}} = \frac{\partial \lambda}{\partial \phi} \frac{d(\phi, \phi_1, \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})} = \frac{d(\lambda, \phi_1, \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})},$$

from which we deduce the theorem :

All the functions isogenous to an elementary function are themselves elementary.

4. If we apply to the elementary functions the formula (2), section 8, relative to the possibility of defining isogenous functions, we have

$$(2) \quad F | [S_r] | = \int_{S_r} \phi \frac{d(\phi_0, \phi_1 \cdots \phi_r)}{d(\omega_1 \cdots \omega_r)} d\omega_1 \cdots d\omega_r,$$

where

$$x_1 = x_1(\omega_1, \cdots \omega_r), \quad x_2 = x_2(\omega_1, \cdots \omega_r), \quad \cdots \quad x_n = x_n(\omega_1, \cdots \omega_r),$$

the equations of the hyperspace S_r .

II. The composition of functions of hyperspaces

I. The results which we have obtained in the preceding section can be expressed in a different form by means of special symbols. That is what we shall do in this section, after having proved a fundamental theorem.

Let $F | [S_r] |$ and $\Phi | [S_{i-r}] |$ be two regular functions of

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hyperspaces, and write

$$\frac{dF}{d(x_{h_1} \dots x_{h_{r+1}})} = p_{h_1 \dots h_{r+1}}, \quad \frac{\partial \phi}{\partial (x_{h_{r+2}} \dots x_{h_{t+2}})} = q_{h_{r+2}} \dots h_{t+2}.$$

$$(I) \quad m_{i_1 \dots i_{t+2}} = \sum_h (-1)^{\binom{h_1 \dots h_{t+2}}{i_1 \dots i_{t+2}}} p_{h_1 \dots h_{r+1}} q_{h_{r+2} \dots h_{t+2}},$$

in which $h_1 \dots h_{t+2}$ is a permutation of $i_1 \dots i_{t+2}$; the sum Σ_h is extended over all the combinations of the $t+2$ subscripts $i_1 \dots i_{t+2}$, $r+1$ at a time; and the symbol $(-1)^{\binom{h_1 \dots h_{t+2}}{i_1 \dots i_{t+2}}}$ represents $+1$ or -1 , according as the substitution which appears in the exponent is even or odd.

2. We shall show that *there exists a regular function* $\Psi[S_{t+1}]$, *such that*

$$\frac{\partial \Psi}{\partial (x_{i_1} \dots x_{i_{t+2}})} = m_{i_1 \dots i_{t+2}}.$$

In fact, the quantities m satisfy the conditions of integrability (section 5, article I); that is,

$$\sum_1^{t+3} (-1)^s \frac{\partial m_{i_1 \dots i_{s-1} i_{s+1} \dots i_{t+3}}}{\partial x_{i_s}} \\ = \sum_1^{t+3} (-1)^s \sum_h (-1)^{\binom{h_1 \dots h_{t+2}}{i_1 \dots i_{s-1} i_{s+1} \dots i_{t+3}}} \frac{\partial}{\partial x_{i_s}} (p_{h_1 \dots h_{r+1}} q_{h_{r+2} \dots h_{t+3}}) = 0.$$

3. To represent the fact that the relation (I) holds among the three functions F , Φ , Ψ we shall write

$$\Psi \equiv (F, \Phi).$$

We have immediately

$$(F, \Phi) \equiv (-1)^{(r+1)(t-r+1)} (\Phi, F).$$

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If $\Theta | [S_{v+1}] |$ is a regular function, and we write

$$\begin{aligned} \frac{\partial \Theta}{\partial (x_{h_{t+3}} \dots x_{h_{v+3}})} &= n_{h_{t+3}} \dots n_{h_{v+3}} \\ l_{i_1 \dots i_{v+3}} &= \sum_h (-1)^{\binom{h_1 \dots h_{v+3}}{i_1 \dots i_{v+3}}} p_{h_1 \dots h_{r+1}} q_{h_{r+2} \dots h_{t+2}} n_{h_{t+3}} \dots n_{h_{v+3}} \\ &= \sum_h (-1)^{\binom{h_1 \dots h_{v+3}}{i_1 \dots i_{v+3}}} m_{h_1 \dots h_{t+2}} n_{h_{t+3}} \dots n_{h_{v+3}}, \end{aligned}$$

it follows that there exists a function $\Lambda | [S_{v+2}] |$ which is regular, and such that

$$\frac{\partial \Lambda}{\partial (x_{i_1} \dots x_{i_{v+3}})} = l_{i_1 \dots i_{v+3}}.$$

We shall write $\Lambda = (F, \Phi, \Theta).$

And in general if the functions $F^{(i)} | [S_r] |$ are regular, we shall understand by

$$(2) \quad M \equiv (F^{(1)}, F^{(2)}, \dots F^{(k)})$$

a regular function of hyperspaces $S_R, R = \sum_1^k r_i + k$, obtained as follows :

$$\Phi_2 \equiv (F^{(1)}, F^{(2)}), \Phi_3 \equiv (\Phi_2, F^{(3)}), \dots M = (\Phi_{k-1}, F^{(k)}).$$

We shall say that M is *composed* of the functions $F^{(1)}, F^{(2)}, \dots F^{(k)}$ and we shall call the operation denoted by (2) the *composition* of the functions $F^{(1)}, F^{(2)}, \dots F^{(k)}$. The operation of *composition* of the functions $F^{(i)}$ evidently possesses the *associative* property. Inversion of the elements of M can only produce changes in sign in the result.

The $F^{(i)}$ will be spoken of as the *divisors* of M . If M has no other divisors but itself, it will be spoken of as *prime*. If two functions have no *divisor* in common, they will be said to be *mutually prime*.

4. Without stopping to develop the theory of divisibility in the present sense, we can give directly a few of its proper-

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ties and apply them to the results of the preceding sections. Thus, every regular function, which is not prime, can be decomposed into prime divisors, and this decomposition can be effected in more than one way. If a function divides one of the divisors of a function, it divides the function itself.

Two functions F and Φ will be isogenous when

$$F \equiv (\Psi, f), \quad \Phi = (\Psi, \phi),$$

where f, ϕ are point functions and f is a function of ϕ . If F and Φ are isogenous, so will be also the functions

$$(F, \Theta) \text{ and } (\Phi, \Theta).$$

No function is isogenous to a prime function; in order that a function may be found isogenous to a given function it is necessary and sufficient that the given function should admit a divisor which is a point function. That is, it is necessary for it to have the form $F \equiv (\Psi, f)$ with f a point function.

An elementary function is obtained by the composition of point functions, etc., etc.

12. *New considerations with reference to the relation of isogeneity*

1. So far we have been considering isogeneity between functions of hyperspaces of the same number of dimensions. We are now to generalize this relation so that it will apply to hyperspaces of different dimensions. Let us consider the two regular functions $\Phi | [S_r]|, \Psi | [S_t]|$, with $r > t$, and write

$$\frac{\partial \Phi}{\partial (x_{t_1} \cdots x_{t_{r+1}})} = a_{t_1 \cdots t_{r+1}} \quad , \quad \frac{\partial \Psi}{\partial (x_{t_1} \cdots x_{t_{t+1}})} = b_{t_1 \cdots t_{t+1}}.$$

We shall say that Φ and Ψ are isogenous when the following conditions are satisfied :

$$(I) \quad \sum_1^{r+2} (-1)^s a_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2}} b_{t_s t_1 \cdots t_t} = 0.$$

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In the case where r is equal to t , these equations imply that the functions not only are isogenous in our first sense, but also that they are elementary. Conversely, if two elementary functions of hyperspaces of the same number of dimensions are isogenous in the sense of section 6, they are also in the present sense.

2. It is easy to show that *every function which admits Φ as divisor is isogenous to Ψ* . In fact, if we take

$$c_{t_1 \dots t_{v+2}} = \sum (-1)^{\binom{h_1 \dots h_{v+2}}{t_1 \dots t_{v+2}}} a_{h_1 \dots h_{r+1}} a'_{r+2, \dots, v+2},$$

we shall have
$$\sum_1^{v+3} (-1)^s c_{t_1 \dots t_{s-1} t_{s+1} \dots t_{v+3}} b_{t_s h_1 \dots h_t} = 0,$$

which proves the theorem.

3. We can now generalize a theorem given in section 7, article 2. We have:

The necessary and sufficient condition that $\phi \mid [S_r] \mid$ shall be isogenous to the elementary function $\Psi \mid [S_{r-1}] \mid$, is that

$$(2) \quad \Phi \mid [S_r] \mid = (\Psi, \Theta).$$

That the condition is sufficient can be shown without any difficulty. In order to show that it is also necessary, let us write

$$\begin{aligned} \frac{d\Phi}{d(x_{t_1} \dots x_{t_{r+1}})} &= a_{t_1 \dots t_{r+1}}, & \frac{d\Psi}{d(x_{t_1} \dots x_{t_{r-t+1}})} &= b_{t_1 \dots t_{r-t+1}}, \\ \frac{d\Theta}{d(x_{t_1} \dots x_{t_t})} &= c_{t_1 \dots t_t}, \\ b_{t_1 \dots t_{r-t+1}} &= \frac{d(\phi_1, \phi_2 \dots \phi_{r-t+1})}{d(x_{t_1}, x_{t_2} \dots x_{t_{r-t+1}})}. \end{aligned}$$

We shall show that if (1) is true, (2) is also true; that is, that

$$(2') \quad a_{t_1 \dots t_{r+1}} = \sum (-1)^{\binom{h_1 \dots h_{r+1}}{t_1 \dots t_{r+1}}} b_{h_1 \dots h_{r-t+1}} c_{h_{r-t+2} \dots h_{r+1}}.$$

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For this purpose let us make a change of variable, taking instead of x_1, x_2, \dots, x_n the new variables $\phi_1, \phi_2, \dots, \phi_{t+1}, x_{t+2}, \dots, x_n$. If we indicate with a prime the symbols that belong with the new variables, we shall have

(i) If $h_{r-t+2}, h_{r-t+3}, \dots, h_r \neq \phi_1, \phi_2, \dots, \phi_{r-t+1}$, then

$$c_{h_{r-t+2} \dots h_r} = c'_{h_{r-t+2} \dots h_r} + \sum (-1)^{\binom{h_{p_1} \dots h_{p_t}}{h_{r-t+2} \dots h_{r+1}}} c'_{h_{p_1} \dots h_{p_{t-1}} l_s} \frac{d(\phi_{l_1} \dots \phi_{l_s})}{d(x_{h_{p_{t-1}+1}} \dots x_{h_{p_t-1}})},$$

in which l_1, \dots, l_s are s of the numbers $1, 2, \dots, r-t+1$, and $h_{p_1} \dots h_{p_t}$ is a permutation of the numbers $h_{r-t+2}, \dots, h_{r+1}$.

(ii) If one of the numbers $h_{r-t+2} \dots h_r$ is equal to one of the numbers $1, 2, \dots, t+1$, then

$$c_{h_{r-t+2} \dots h_r} = \sum (-1)^{\binom{h_{p_1} \dots h_{p_{t-1}}}{h_{r-t+2} \dots h_r}} c'_{h_{p_1} \dots h_{p_{t-1}} l_s} \frac{d(\phi_{l_1} \dots \phi_{l_s})}{d(x_{h_{p_{t-1}+1}} \dots x_{h_{p_t-1}})}$$

Equation (2') will then become

$$\begin{aligned} (2'') \quad a_{i_1 \dots i_{r+1}} &= \sum (-1)^{\binom{h_1 \dots h_{r+1}}{i_1 \dots i_{r+1}}} b_{h_1 \dots h_{r-t+1}} c'_{h_{r-t+2} \dots h_{r+1}} \\ &+ \sum (-1)^{\binom{h_1 \dots h_{r+1}}{i_1 \dots i_{r+1}}} \frac{d(\phi_1 \dots \phi_{r-t+1})}{d(x_{h_1} \dots x_{h_{r-t+1}})} \\ &\quad \sum (-1)^{\binom{h_{p_1} \dots h_{p_t}}{h_{r-t+2} \dots h_{r+1}}} c'_{h_{p_1} \dots h_{p_{t-1}} l_s} \frac{d(\phi_{l_1} \dots \phi_{l_s})}{d(x_{h_{p_{t-1}+1}} \dots x_{h_{p_t-1}})}, \end{aligned}$$

in which the first sum is extended over all the possible combinations of the indices $h_{r-t+2} \dots h_{r+1}$ which do not contain any of the numbers $1, 2, \dots, r-t+1$. The second sum may be rewritten in the form

$$\begin{aligned} &\sum (-1)^{\binom{h_{p_1} \dots h_{p_{r+1}}}{i_1 \dots i_{r+1}}} c'_{h_{p_1} \dots h_{p_{r-t+1}} l_s} \\ &\sum (-1)^{h_{p_t}} \frac{d(\phi_1 \dots \phi_{r-t+1})}{d(x_{h_1} \dots x_{h_{r-t+1}})} \frac{d(\phi_{l_1} \dots \phi_{l_s})}{d(x_{h_{p_{t-1}+1}} \dots x_{h_{p_t-1}})}, \end{aligned}$$

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whence it vanishes. The equation (2'') reduces then to

$$(2''') \quad a_{t_1 \dots t_{r+1}} = \sum (-1)^{\binom{h_1 \dots h_{r+1}}{t_1 \dots t_{r+1}}} b_{h_1 \dots h_{r-t+1}} c'_{h_{r-t+2} \dots h_{r+1}}.$$

In particular we have

$$a_{1, 2, \dots, t+1, t_{t+2} \dots t_{r+1}} = b_{1, 2, \dots, t+1} c'_{t_{t+2} \dots t_{r+1}}$$

so that

$$(3) \quad c'_{t_{t+2} \dots t_{r+1}} = \frac{a_{1, 2, \dots, t+1, t_{t+2} \dots t_{r+1}}}{\left\{ \frac{d(\phi_1 \dots \phi_{t+1})}{d(x_1 \dots x_{t+1})} \right\}}.$$

Now by following a process analogous to that of section 7, article 2, it is easy to show that all the equations (2''') are a consequence of these last equations (3). And so it is sufficient for us to show that the quantities c' , obtained from (3), satisfy the conditions of integrability. We have in fact

$$a'_{1, \dots, t+1, t_{t+2}, \dots, t_{r+1}} = \frac{a_{1, \dots, t+1, t_{t+2}, \dots, t_{r+1}}}{\frac{d(\phi_1 \dots \phi_{t+1})}{d(x_1 \dots x_{t+1})}},$$

while a' will be zero if it has less than $t+1$ of its subscripts taken from the numbers $1, 2, \dots, t+1$. If we apply then a process of reasoning analogous to that of section 7, article 2, we find that the conditions of integrability will be satisfied for the quantities c' .

13. Differentiation and integration

I. If two functions $F | [S_n] |$, $\Phi | [S_r] |$ are regular and isogenous, we know that the ratio

$$\phi = \frac{\left(\frac{d\Phi}{dS_{r+1}} \right)}{\left(\frac{dF}{dS_{r-1}} \right)}$$

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will be independent of the hyperspace S_{r+1} , and will depend merely upon the point of the space at which the derivative is taken. The quantity ϕ will then be a point function for the total space of n dimensions. We shall denote it with the symbol $\frac{d\Phi}{dF}$ and call it the *derivative of Φ with respect to F* .

As a fundamental theorem it can be shown that the *derivative of Φ with respect to F is isogenous to both of the functions Φ and F* . The proof of this theorem comes immediately from formula (1) of section 7, with reference to the definition given in the preceding section.

2. Consider now a point function f isogenous to a regular function $F \mid [S_r] \mid$. By fixing the direction of the hyperspace S_{r+1} (see section 1, article 2) the quantity $\frac{dF}{dS_{r+1}}$ will be defined (see section 3, article 7), and hence the quantity

$$\int_{S_{r+1}} f \frac{dF}{dS_{r+1}} dS_{r+1}$$

will also be defined. This integral we shall represent by the symbol

$$\int_{S_{r+1}} f dF.$$

Changing the direction of the hyperspace will change the sign of the integral.

We shall suppose that the hyperspace S_{r+1} is closed and forms the boundary of a hyperspace S_{r+2} immersed in a portion of the total hyperspace S_n throughout which f and F have no singularities. It follows that

$$\begin{aligned} \int_{S_{r+1}} f dF &= \int_{S_{r+1}} f \sum \frac{dF}{d(x_{t_1} \dots x_{t_{r+1}})} \alpha_{t_1 \dots t_{r+1}} dS_{r+1} \\ &= \int_{S_{r+1}} f \sum p_{t_1 \dots t_{r+1}} \alpha_{t_1 \dots t_{r+1}} dS_{r+1}, \end{aligned}$$

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where the $\alpha_{i_1 \dots i_{r+1}}$ are the direction cosines of the hyperspace S_{r+1} . If we choose properly the direction of the hyperspace S_{r+2} and apply the generalization of Stokes's theorem (see section 4) we shall have

$$\begin{aligned} \int_{S_{r+1}} f dF &= \int_{S_{r+2}} \sum \beta_{i_1 \dots i_{r+2}} \sum (-1)^{s-1} \frac{\partial (f p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}})}{\partial x_{i_s}} dS_{r+2} \\ &= \int_{S_{r+2}} \sum \beta_{i_1 \dots i_{r+2}} \left\{ \sum (-1)^s p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}} \frac{\partial f}{\partial x_{i_s}} \right. \\ &\quad \left. + f \sum (-1)^{s-1} \frac{\partial p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+2}}}{\partial x_{i_s}} \right\} dS_{r+2} = 0. \end{aligned}$$

Hence we have the theorem expressed by the formula

$$(I) \quad \int_{S_{r+1}} f dF = 0.$$

If, instead of a single hyperspace S_{r+1} we have the hyperspaces $S_{r+1}^{(i)}$ ($i = 1, 2, \dots, n$) which bound a space S_{r+2} within which there are no singularities for f or F , we shall have the formula :

$$(I') \quad \sum_1^n \int_{S_{r+1}^{(i)}} f dF = 0,$$

in which the directions of the hyperspaces $S_{r+1}^{(i)}$ are all to be chosen with reference to the conventions adopted for the generalization of Stokes's theorem.

The theorem enunciated in the formulæ (I) and (I') is the direct extension of Cauchy's theorem.

3. Let us take away from the total hyperspace all those portions in which either f or F have singularities, and then introduce cuts in such a way that every closed hyperspace S_{r+1} may be taken as the complete boundary of a hyperspace S_{r+2} .

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Take two hyperspaces S_r^0 , S_r' such that a hyperspace S_{r+1} can be drawn to have them for its boundary, and choose the positive direction of S_r^0 and the negative direction of S_r' so as to correspond by the theorem of Stokes to one direction of the hyperspace S_{r+1} . With the direction of S_{r+1} fixed in this way, the integral

$$(2) \quad \int_{S_{r+1}} f dF$$

will be determined.

It is easy to show that the value of the integral (2) will not depend on the hyperspace S_{r+1} , but merely on S_r^0 and S_r' . In fact if S_{r+1}' is another hyperspace which has these same two spaces for its boundary, the totality of S_{r+1} and S_{r+1}' will form a closed hyperspace, and from the hypotheses that we have made, we shall have

$$\int_{S_{r+1} + S_{r+1}'} f dF = 0,$$

from which the desired property follows.

Therefore the integral (2) can be indicated by the expression

$$(2') \quad \int_{S_r^0}^{S_r'} f dF.$$

By changing the direction of S_{r+1} we change the sign of the integral; hence we may write

$$(3) \quad \int_{S_r^0}^{S_r'} f dF = - \int_{S_r'}^{S_r^0} f dF.$$

4. If we keep fixed the hyperspace S_r^0 and vary S_r' , the integral (2') may be regarded as a function (regular) of S_r' , and we can write

$$(4) \quad \int_{S_r^0}^{S_r'} f dF = \Phi | [S_r'] |.$$

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The function Φ will be isogenous to F and we shall have

$$(5) \quad \frac{d\Phi}{dF} = f,$$

that is to say, the two operations of integration and differentiation are mutually inverse.

14. Isogeneity of order r

1. A system of elementary functions will be said to have isogeneity of order r when all the functions of order greater than or equal to r , which are obtained from the system by means of composition (see section 11), vanish, while there is at least one function of order $r - 1$ which does not vanish. All the elementary functions $\Phi | [S_i]$ of the system must depend on certain functions $\phi_1, \phi_2, \dots, \phi_k, \dots$ in such a way (see section 10) that

$$\frac{\partial \Phi}{\partial (x_{i_1} \dots x_{i_{r+1}})} = \frac{d(\phi_{i_1} \dots \phi_{i_{r+1}})}{d(x_{i_1} \dots x_{i_{r+1}})}, \quad \Phi \equiv (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_{r+1}}).$$

2. We have immediately the following theorems:

The necessary and sufficient condition for isogeneity of order r that is

$$(1) \quad \frac{d(\phi_{i_1} \dots \phi_{i_{r+1}})}{d(x_{i_1} \dots x_{i_{r+1}})} = 0$$

for every possible combination of the numbers $l_1, \dots, l_{r+1}, i_1, \dots, i_{r+1}$.

A function of order $r - 1$ is always isogenous to any other function of the system.

In fact from (1) it follows that every function of order $r - 1$ is isogenous to the functions of order zero of the system, that is, to the functions ϕ_i . We shall have

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then

$$\begin{aligned} q_{i_s h_1 \dots h_t} &= \frac{\partial \Phi}{\partial (x_{i_s} x_{h_1} \dots x_{h_t})} = \sum_1^{t+1} (-1)^{u-1} \frac{\partial \phi_{i_u}}{\partial x_{i_s}} \frac{d(\phi_{i_1} \dots \phi_{i_{u-1}} \phi_{i_{u+1}} \phi_{i_{t+1}})}{d(x_{h_1} \dots x_{h_t})} \\ &= \sum_1^{t+1} (-1)^{u-1} \frac{\partial \phi_{i_u}}{\partial x_{i_s}} N_u. \end{aligned}$$

And if we let $\psi \mid [S_{r-1}]$ represent one of the functions of order $r - 1$ of the system, and write

$$\frac{\partial \psi}{\partial (x_{i_1} \dots x_{i_r})} = p_{i_1 \dots i_r},$$

we shall have

$$\begin{aligned} \sum_1^{r+1} (-1)^s p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}} q_{i_s h_1 \dots h_t} \\ = \sum_1^{t+1} (-1)^{u-1} N_u \sum_1^{r+1} (-1)^s p_{i_1 \dots i_{s-1} i_{s+1} \dots i_{r+1}} \frac{\partial \phi_{i_u}}{\partial x_{i_s}}. \end{aligned}$$

Every function of order $r - 1$ admits as divisor another function of the system of lower order (see section 11, article 3).

3. Let us consider specially the functions of the system of order zero; that is, the functions $\phi_1, \phi_2, \dots, \phi_r, \dots$. By means of the equations (1) we know that *there must be r of them, $\phi_1, \phi_2, \dots, \phi_r$, independent, of which all the others are functions*, and conversely, that *every function of $\phi_1, \phi_2, \dots, \phi_r$, will be an elementary function in the system, and will be of order zero*.

If we take two functions Φ and F of order $r - 1$, they will be isogenous, and we shall have the relation

$$(2) \quad \frac{d\Phi}{dF} = \phi(\phi_1, \phi_2, \dots, \phi_r).$$

Further, if we take an arbitrary function ϕ of order zero, that is a function of $\phi_1, \phi_2, \dots, \phi_r$, we shall have

$$(3) \quad \int_{S_r} \phi dF = 0,$$

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where S_r is the complete boundary of a space S_{r+1} within which ϕ and F have no singularities. If we have

$$F \equiv (\phi_1, \phi_2, \dots \phi_r),$$

then (3) can be written in the form

$$\int_{S_r} \phi \frac{d(\phi_1, \phi_2, \dots \phi_r)}{d(\omega_1, \omega_2, \dots \omega_r)} d\omega_1, d\omega_2, \dots d\omega_r = 0,$$

$\omega_1, \dots \omega_r$ being the parameters of the hyperspace S_r (see section 1, articles 1, 2). If we take

$$\frac{d\phi_i}{d\omega_s} d\omega_s = d_s \phi_i,$$

we shall have
$$\int_{S_r} \phi \begin{vmatrix} d_1 \phi_1 & d_2 \phi_1 & \dots & d_r \phi_1 \\ d_1 \phi_2 & d_2 \phi_2 & \dots & d_r \phi_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_1 \phi_r & d_2 \phi_r & \dots & d_r \phi_r \end{vmatrix} = 0,$$

which is but a *generalization of Cauchy's theorem* (see the preceding section) put in a different form for the case of the elementary functions.

If S_r is not closed, but is bounded by two hyperspaces S_{r-1}^0 and S_{r-1} , of which the first is fixed and the second variable, we shall have defined the expression

$$\Phi [S_{r-1}] = \int_{S_{r-1}^0}^{S_{r-1}} \phi \begin{vmatrix} d_1 \phi_1 & \dots & d_r \phi_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_1 \phi_r & \dots & d_r \phi_r \end{vmatrix}.$$

Third Lecture

ON THE THEORY OF WAVES AND GREEN'S METHOD*

SECTION I

LET a homogeneous liquid be subjected to certain forces and let it occupy a domain S . Let this domain be limited by a frontier σ which is composed partly of a set ω' of rigid boundaries, and partly of a free surface ω , where the pressure is P .

Let us suppose that the state of equilibrium is stable. We shall study the small oscillations of the fluid when it is displaced from the state of equilibrium.

The hydrodynamical equations of Lagrange are

$$\left. \begin{aligned} \frac{d^2x}{dt^2} \cdot \frac{\partial x}{\partial x_0} + \frac{d^2y}{dt^2} \cdot \frac{\partial y}{\partial x_0} + \frac{d^2z}{dt^2} \cdot \frac{\partial z}{\partial x_0} &= \frac{\partial}{\partial x_0} \left(V - \frac{P}{\rho} \right) \\ \frac{d^2x}{dt^2} \cdot \frac{\partial x}{\partial y_0} + \frac{d^2y}{dt^2} \cdot \frac{\partial y}{\partial y_0} + \frac{d^2z}{dt^2} \cdot \frac{\partial z}{\partial y_0} &= \frac{\partial}{\partial y_0} \left(V - \frac{P}{\rho} \right) \\ \frac{d^2x}{dt^2} \cdot \frac{\partial x}{\partial z_0} + \frac{d^2y}{dt^2} \cdot \frac{\partial y}{\partial z_0} + \frac{d^2z}{dt^2} \cdot \frac{\partial z}{\partial z_0} &= \frac{\partial}{\partial z_0} \left(V - \frac{P}{\rho} \right) \end{aligned} \right\} \quad (1)$$

where x, y, z , denote the coördinates of points of the fluid at time t , x_0, y_0, z_0 the initial coördinates, V the potential function, P the pressure, ρ the density.

2. Let x_0, y_0, z_0 be the coördinates which correspond to the state of stable equilibrium, ξ, η, ζ the components of displacement of each particle with respect to its position of equilibrium.

Then $x = x_0 + \xi, \quad y = y_0 + \eta, \quad z = z_0 + \zeta.$

* Translated from the French by Professor Percy John Daniell, of the Rice Institute.

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If we consider the displacements as infinitesimals of the first order and if we neglect terms of order higher than the first, the equations (1) become

$$\left. \begin{aligned} \frac{d^2\xi}{dt^2} &= \frac{\partial}{\partial x_0} \left(V - \frac{P}{\rho} \right) \\ \frac{d^2\eta}{dt^2} &= \frac{\partial}{\partial y_0} \left(V - \frac{P}{\rho} \right) \\ \frac{d^2\zeta}{dt^2} &= \frac{\partial}{\partial z_0} \left(V - \frac{P}{\rho} \right) \end{aligned} \right\}$$

For simplification the indices 0 are suppressed and x, y, z denote the coördinates of each particle in the position of equilibrium.

$$\text{Then} \quad \left. \begin{aligned} \frac{d^2\xi}{dt^2} &= \frac{\partial}{\partial x} \left(V - \frac{P}{\rho} \right) \\ \frac{d^2\eta}{dt^2} &= \frac{\partial}{\partial y} \left(V - \frac{P}{\rho} \right) \\ \frac{d^2\zeta}{dt^2} &= \frac{\partial}{\partial z} \left(V - \frac{P}{\rho} \right) \end{aligned} \right\} \quad (2)$$

The condition of incompressibility can be written as

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0. \quad (3)$$

On account of (2) we can put

$$\xi = \frac{\partial \Phi}{\partial x}, \quad \eta = \frac{\partial \Phi}{\partial y}, \quad \zeta = \frac{\partial \Phi}{\partial z},$$

Φ being the potential of displacement.

Then the equations (2) become

$$\frac{d^2\Phi}{dt^2} - V + \frac{P}{\rho} = c, \quad (4)$$

where c is constant with respect to x, y, z , but may vary with t .

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The equation (3) becomes

$$\Delta^2\Phi = 0.$$

At points of the liquid where it touches the rigid boundary

$$\xi \cos nx + \eta \cos ny + \zeta \cos nz = 0,$$

if n denotes the normal to the boundary.

This condition becomes $\frac{\partial\Phi}{\partial n} = 0$.

3. Let us return to the equation (4). If we put

$$V - \frac{P}{\rho} + c = H,$$

the equation (4) becomes $\frac{d^2\Phi}{dt^2} = H$. (4')

The free surface of the fluid has been denoted by ω . Let us suppose that the potential function V and the pressure P , which correspond to each particle of fluid belonging to ω are functions of the coördinates of the point occupied by the particle independently of the form of the liquid. If this hypothesis is not correct, since the displacements are infinitesimal, we can neglect the variations produced by the changes in form of the fluid so that we can always proceed as if the hypothesis were correct.

In the state of equilibrium H is constant on ω . Therefore the equation of this surface will be

$$H = H_0 = \text{constant}.$$

Let us now calculate H when a point of the surface ω is displaced when ξ, η, ζ are the components of displacement.

If we neglect infinitesimals of a higher order than the first,

$$H = H_0 + \frac{\partial H}{\partial x}\xi + \frac{\partial H}{\partial y}\eta + \frac{\partial H}{\partial z}\zeta.$$

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Then putting $\lambda^2 = \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2$,

$$\frac{\partial H}{\partial x} = \lambda \cos nx, \quad \frac{\partial H}{\partial y} = \lambda \cos ny, \quad \frac{\partial H}{\partial z} = \lambda \cos nz, \quad (5)$$

when n is the normal to the surface ω .

$$\begin{aligned} \text{Then} \quad H &= H_0 + \lambda (\xi \cos nx + \eta \cos ny + \zeta \cos nz) \\ &= H_0 + \lambda \frac{\partial \Phi}{\partial n}; \end{aligned}$$

combining this with equation (4')

$$\frac{\partial^2 \Phi}{\partial t^2} = H_0 + \lambda \frac{\partial \Phi}{\partial n}$$

or
$$\frac{\partial^2 \Phi}{\partial t^2} = \lambda \frac{\partial \Phi}{\partial n},$$

since Φ is determinate except for a quantity which is constant with respect to the time.

Let us take the normal n as directed toward the interior of the fluid, and let us suppose that $V - \frac{P}{\rho}$ increases on moving ω and following the positive direction of n .

Then when n is positive, $\frac{\partial H}{\partial n} > 0$,

or by virtue of the equations (5)

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial x} \cos nx + \frac{\partial H}{\partial y} \cos ny + \frac{\partial H}{\partial z} \cos nz = \lambda,$$

it follows that $\lambda > 0$.

The problem of waves can be presented in the following manner.

4. To determine a function Φ regular within the domain S which satisfies the equation

$$(A) \quad \Delta^2 \Phi = 0$$

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within S and which in the part ω' of the boundary satisfies the condition

$$(B) \quad \frac{\partial \Phi}{\partial n} = 0$$

and in the part ω satisfies the condition

$$(C) \quad \frac{\partial^2 \Phi}{\partial t^2} = \lambda \frac{\partial \Phi}{\partial n},$$

where λ is a positive quantity independent of the time, and n is the normal to the boundary directed toward the interior of the domain S .

SECTION 2

1. We can make a comparison between the problem we are about to consider and that of the vibrations of elastic media, and other problems of mathematical physics. The problem of the vibrations of elastic media is based upon the equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \Delta^2 u. \quad (6)$$

The problem of the propagation of heat in the case of varying temperature leads to the equation

$$\frac{\partial V}{\partial t} = a \Delta^2 V. \quad (7)$$

The problems of potential and of stationary temperatures in isotropic bodies depend upon the equation of Laplace

$$\Delta^2 W = 0. \quad (8)$$

These three equations are respectively of *hyperbolic*, *parabolic*, and *elliptic* types.

The question we have considered in section 1 belongs to the elliptic type on account of the equation (A) of section 1, which is the equation of Laplace; but it is the condition which must be satisfied on the surface ω of the boundary

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which leads to the essential difference between this problem and the problems of potential and stationary temperatures. In fact, in the problems of potential the conditions at the boundary are reduced to that of giving the values of the unknown function or of its normal derivative; in those of stationary temperatures a linear relation between the unknown function and its normal derivative is known. But in the case of waves the condition at the boundary (equation (C) of section 1) introduces a new variable, the time, which makes the problem one of four variables. In respect to the number of variables the problem of waves is similar to the problems of vibrations and varying temperatures. It differs from them, however, because equations (6) and (7) have real characteristics. There are no real characteristics in the problem of the waves of liquids. We shall give a theorem in section 3 which will show the difference, from a physical standpoint, between waves in elastic media and waves in liquids.

2. There are two general methods in which the different problems we are investigating can be treated.

That of the separation of variables consists in separating the time from the space variables.

Let us put in the equation (6)

$$U = \sin mt \cdot u(x, y, z), \quad (9)$$

where m is a constant.

The equation becomes

$$m^2 u + \alpha^2 \Delta^2 u = 0, \quad (10)$$

where the time has disappeared. If, for example, on the boundary $U = 0$, u must be taken $= 0$ there. We are led to find values of m for which the previous equation has solutions which are not identically zero (special solutions). The general solution is obtained by forming an infinite series of

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solutions of the form (9) multiplied by arbitrary constants of such values that U and $\frac{\partial U}{\partial t}$ for $t = 0$ have the values of the given functions of x, y, z .

The question of determining the special solutions has been resolved by Poincaré; the theory of integral equations has been used and Mr. Hilbert, Mr. Schmidt, and others have founded the theory of series of special solutions.

Similarly an analogous process can be employed for equation (7) if we put $V = e^{mt}v(x, y, z)$; that is to say, by separating the time from the variables x, y, z .

Equation (7) reduces then to

$$mv + a\Delta^2v = 0,$$

which is exactly analogous to equation (10).

3. The same method of the separation of the variables can be applied to the problem of waves in liquids.

If we put $\Phi = \sin mt \phi(x, y, z)$ equation (A) of section 1 becomes

$$\Delta^2\phi = 0,$$

equation (B) is
$$\frac{\partial\phi}{\partial u} = 0,$$

and equation (C) must be replaced by

$$m^2\phi + \lambda \frac{\partial\phi}{\partial n} = 0.$$

Here again the values of m corresponding to solutions ϕ which are not identically zero (special solutions) must be found.

By series of special solutions the general solution can be obtained. To calculate the values of m the method of Poincaré with those of integral equations can be used.

4. But we wish to set aside the process of the separation of variables and to pass on to the other general method. It

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is the method which is connected with the ideas which Green used for the first time for the equation of Laplace and which, little by little, has been also used for other types of equations. By this point of view Kirchhoff arrived at his celebrated formula which expresses the principle of Huyghens. He applied Green's method to equation (6).

Betti has also applied an analogous method to equation (7).

We wish to show that a general formula can be found in the case of waves of fluids of a type which presents some analogies to these formulæ. I have had occasion to mention this formula without giving any development from it in my lectures at Stockholm. We shall now develop it and demonstrate in detail some applications of it.

SECTION 3

1. We shall begin by demonstrating in this paragraph some general theorems.

First Theorem. If Φ is the function which satisfies the conditions (A), (B), (C) of section 1, it is determinate if the values $\Phi_0, \left(\frac{\partial \Phi}{\partial t}\right)_0$ of Φ and $\left(\frac{\partial \Phi}{\partial t}\right)$ for $t = 0$ on the surface ω are known.

Demonstration. Let Φ_1, Φ_2 be two functions which satisfy the conditions to which Φ is subjected.

Their difference $\Phi_3 = \Phi_1 - \Phi_2$ also satisfies the equations (A), (B), (C) and further we have

$$(\Phi_3)_0 = 0 \qquad \left(\frac{\partial \Phi_3}{\partial t}\right)_0 = 0$$

for $t = 0$ on the surface ω .

Let us now calculate

$$\Omega = \frac{1}{2} \frac{\partial}{\partial t} \int_{\omega} \frac{1}{\lambda} \left(\frac{\partial \Phi_3}{\partial t}\right)^2 d\omega.$$

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On account of equation (C) we shall have

$$\Omega = \int_{\omega} \frac{1}{\lambda} \left(\frac{\partial \Phi_3}{\partial t} \right) \left(\frac{\partial^2 \Phi_3}{\partial t^2} \right) d\omega = \int_{\omega} \left(\frac{\partial \Phi_3}{\partial t} \right) \left(\frac{\partial \Phi_3}{\partial n} \right) d\omega.$$

But on ω' $\frac{\partial \Phi_3}{\partial n} = 0$ and therefore

$$\Omega = \int_{\sigma} \left(\frac{\partial \Phi_3}{\partial t} \right) \left(\frac{\partial \Phi_3}{\partial n} \right) d\sigma.$$

Applying a well-known transformation,

$$\begin{aligned} -\Omega = \int_S \left(\frac{\partial}{\partial x} \frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial y} + \frac{\partial}{\partial z} \frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial z} \right) dS \\ + \int_S \frac{\partial \Phi_3}{\partial t} \Delta^2 \Phi_3 dS. \end{aligned}$$

The third term = 0; then

$$-\Omega = \frac{1}{2} \frac{\partial}{\partial t} \int_S \left\{ \left(\frac{\partial \Phi_3}{\partial x} \right)^2 + \left(\frac{\partial \Phi_3}{\partial y} \right)^2 + \left(\frac{\partial \Phi_3}{\partial z} \right)^2 \right\} dS$$

and it follows that

$$\frac{1}{2} \frac{\partial}{\partial t} \left[\int_{\omega} \frac{1}{\lambda} \left(\frac{\partial \Phi_3}{\partial t} \right)^2 d\omega + \int_S \left\{ \left(\frac{\partial \Phi_3}{\partial x} \right)^2 + \left(\frac{\partial \Phi_3}{\partial y} \right)^2 + \left(\frac{\partial \Phi_3}{\partial z} \right)^2 \right\} dS \right] = 0.$$

Integrating with respect to the time,

$$\int_{\omega} \frac{1}{\lambda} \left(\frac{\partial \Phi_3}{\partial t} \right)^2 d\omega + \int_S \left\{ \left(\frac{\partial \Phi_3}{\partial x} \right)^2 + \left(\frac{\partial \Phi_3}{\partial y} \right)^2 + \left(\frac{\partial \Phi_3}{\partial z} \right)^2 \right\} dS = c, \quad (11)$$

where c is constant with respect to the time.

Then if $(\Phi_3)_0 = 0$ for $t = 0$ on ω , since $\frac{\partial \Phi_3}{\partial n} = 0$ on ω' $(\Phi_3)_0$ must be zero in the domain S . Consequently, the second integral in the formula (11) will be 0 for $t = 0$. In the same way, since $\left(\frac{\partial \Phi_3}{\partial t} \right)_0 = 0$, the first integral will be 0 for $t = 0$. It follows that $c = 0$, and the conclusion can be drawn that Φ_3 will be 0 for every value of t and therefore $\Phi_1 = \Phi_2$.

Q. E. D.

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2. *Second Theorem.* If at a certain instant the molecules belonging to a part of the domain S are not displaced from the position of equilibrium, any molecule of the fluid is not displaced from the position of equilibrium.

Demonstration. If ξ, η, ζ are 0 in any part of S , Φ will be constant in this part, and since it is an harmonic function regular in S , it will be everywhere constant. Consequently ξ, η, ζ will be 0 at all points of S . Q. E. D.

Third Theorem. If at a certain instant the molecules belonging to a part of the domain S are not displaced from the position of equilibrium and have no velocity, the fluid will remain always in the position of equilibrium.

Demonstration. If ξ, η, ζ and $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$ are 0 in one part of the domain S at a certain instant, Φ and $\frac{d\Phi}{dt}$ will be constant in this part and therefore they will be constant in the whole domain S at the same instant. By virtue of the first theorem they will be constant in S at every instant and consequently the liquid will have no motion. Q. E. D.

3. These propositions show us the essential difference which exists between waves in liquids and waves in elastic media. In elastic media the motion is propagated with a certain velocity from one part to another; in liquids the motion reaches the whole mass contemporaneously, at least when the fluid does not remain in a constant state of equilibrium. In the case of liquids there is no propagation of motion and consequently one cannot speak of the velocity of propagation.

SECTION 4

1. Let Φ and Ψ be two functions which satisfy the conditions (A), (B), (C) of section 1.

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By virtue of Green's theorem

$$\int_{\sigma} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\sigma = 0$$

on account of (B) $\int_{\omega} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\omega = 0.$

Using (C) this becomes

$$\int_{\omega} \left(\Phi \left(\frac{\partial^2 \psi}{\partial t^2} - \Psi \frac{\partial^2 \phi}{\partial t^2} \right) \frac{1}{\lambda} d\omega = 0. \quad (12)$$

Let us now suppose that

$$\Psi = \frac{1}{r} + \chi,$$

where r denotes the distance between a point A (x_0, y_0, z_0) interior to the domain S and a point (x, y, z) and where χ is a regular function. Then the preceding formulæ are no longer valid for they presuppose that ψ is regular in the domain S . In this case formula (12) must be replaced by

$$4 \pi \Phi_A + \int_{\omega} \left(\Phi \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \Phi}{\partial t^2} \right) \frac{1}{\lambda} d\omega = 0, \quad (12')$$

where Φ_A denotes the value of Φ at the point A .

Then $4 \pi \phi_A = - \frac{\partial}{\partial t} \int_{\omega} \left(\phi \frac{\partial \psi}{\partial t} - \psi \frac{\partial \phi}{\partial t} \right) \frac{1}{\lambda} d\omega.$

Integrating between the limits 0 and t_1 , we obtain

$$\begin{aligned} 4 \pi \int_0^{t_1} \phi_A dt &= - \int_{\omega} \left\{ \phi_1 \left(\frac{\partial \psi}{\partial t} \right)_1 - \psi_1 \left(\frac{\partial \phi}{\partial t} \right)_1 \right\} \frac{1}{\lambda} d\omega \\ &+ \int_{\omega} \left\{ \phi_0 \left(\frac{\partial \psi}{\partial t} \right)_0 - \psi_0 \left(\frac{\partial \phi}{\partial t} \right)_0 \right\} \frac{1}{\lambda} d\omega \end{aligned}$$

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where $\phi_1, \psi_1 \left(\frac{\partial \phi}{\partial t} \right)_1 \left(\frac{\partial \psi}{\partial t} \right)_1$ denote the functions ϕ, ψ and the derivatives $\frac{\partial \phi}{\partial t} \frac{\partial \psi}{\partial t}$ for $t=t_1$, while $\phi_0, \psi_0 \left(\frac{\partial \phi}{\partial t} \right)_0 \left(\frac{\partial \psi}{\partial t} \right)_0$ denote the same quantities for $t=t_0$. Let us now suppose that ψ_1 and $\left(\frac{d\psi}{dt} \right)_1$ are 0 on ω .

Then

$$(D) \quad \Phi(x_0, y_0, z_0, t_1) = \frac{1}{4\pi} \frac{d}{dt_1} \int_{\omega} \left\{ \phi_0 \left(\frac{\partial \psi}{\partial t} \right)_0 - \psi_0 \left(\frac{\partial \phi}{\partial t} \right)_0 \right\} \frac{1}{\lambda} d\omega.$$

The above formula gives us a knowledge of Φ at every point in S and for every value of t when the values of $\phi_0, \left(\frac{\partial \phi}{\partial t} \right)_0$ are known on ω . (Compare with the first theorem of section 3.)

It is necessary to calculate further the function Ψ and consequently χ . This function plays, in this case, *a part which can be compared with that played by Green's function.*

It must be remarked that ψ_0 and $\left(\frac{d\psi}{dt} \right)_0$ should depend on t_1 since ψ_1 and $\left(\frac{d\psi}{dt} \right)_1$ should be 0. The variable t_1 appears then in the second member of the equation (D) because it is contained in ψ_0 and $\left(\frac{d\psi}{dt} \right)_0$.

SECTION 5

In this paragraph we shall give some applications of the fundamental formula (D) of the preceding paragraph. Let us suppose that S is a sphere of radius R and that ω is the surface of the sphere in such a way that there are no rigid boundaries.

Let us put

$$\psi = a_0 + \frac{(t_1 - t)^2}{2!} a_2 + \frac{(t_1 - t)^4}{4!} a_4 + \dots,$$

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$a_0, a_2, a_4 \dots$ being coefficients independent of t_1 and t . We shall have

$$\psi_1 = a_0, \quad \left(\frac{d\psi}{dt} \right)_1 = 0.$$

But

$$\psi = \frac{1}{r} + \chi.$$

$$\therefore a_0 = \frac{1}{r_A} + (\chi)_1,$$

and since a_0 should be 0 on ω and χ should be a regular and harmonic function if we use the method of images we obtain

$$(\chi)_1 = -\frac{R}{l} \frac{1}{r_{A'}},$$

where A' denotes the image point of A with respect to the sphere, $r_{A'}$ is the distance of the point A' from the point (x, y, z) , l is the distance from the center of the sphere to the point A .

Then

$$a_0 = \frac{1}{r_A} - \frac{R}{l} \frac{1}{r_{A'}}.$$

Let ρ be the radius vector, the pole being at the center of the sphere; then

$$\frac{\partial \psi}{\partial n} = -\frac{\partial \psi}{\partial \rho} = -\frac{\partial a_0}{\partial \rho} - \frac{(t_1 - t)^2}{2!} \frac{\partial a_2}{\partial \rho} - \frac{(t_1 - t)^4}{4!} \frac{\partial a_4}{\partial \rho} \dots$$

$$\frac{\partial^2 \psi}{\partial t^2} = a_2 + \frac{(t_1 - t)^2}{2!} a_4 + \dots$$

Consequently on the surface ω , i.e. for $\rho = R$

$$-\lambda \frac{\partial a_0}{\partial \rho} = a_2, \quad -\lambda \frac{\partial a_2}{\partial \rho} = a_4, \quad -\lambda \frac{\partial a_4}{\partial \rho} = a_6, \dots$$

Since a_0 is known, the regular harmonic functions $a_2, a_4, a_6 \dots$ must be determinate when their values on the boundary of the sphere are known.

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Let us begin by transforming the expression for a_0 . Let us denote by γ the angle between the lines joining the center of the sphere to the points A and (x, y, z) .

$$\text{Then } a_0 = \frac{1}{(l^2 + \rho^2 - 2 l \rho \cos \gamma)^{\frac{1}{2}}} - \frac{R}{l} \frac{1}{\left(\frac{R^4}{l^2} + \rho^2 - 2 \frac{R^2}{l} \rho \cos \gamma\right)^{\frac{1}{2}}}$$

$$\text{or } \frac{\rho}{R} \frac{\partial a_0}{\partial \rho} = \frac{\rho}{R} \frac{\partial}{\partial \rho} \left[\frac{1}{(l^2 + \rho^2 - 2 l \rho \cos \gamma)^{\frac{1}{2}}} \right] \\ - \frac{\rho}{R} \frac{\partial}{\partial \rho} \left[\frac{R}{l} \cdot \frac{1}{\left(\frac{R^4}{l^2} + \rho^2 - 2 \frac{R^2}{l} \rho \cos \gamma\right)^{\frac{1}{2}}} \right]$$

is a harmonic function which is equal to $\frac{\partial a_0}{\partial \rho}$ on the surface of the sphere; but it is not regular in the interior of the sphere. In fact, the first term of the second member becomes infinite for $\rho = l$, $\gamma = 0$. Then to calculate a_2 we cannot take the previous expression and multiply it by $-\lambda$ for a_2 must be regular in the interior of the sphere. But the following artifice may be used to calculate a_2 .

Let us transform the first term of the second member by a transformation of reciprocal radii with respect to the sphere and let us multiply by $\frac{R}{\rho}$. The expression remains harmonic, possesses the same values on the boundary of the sphere, but becomes regular in the interior. To make the transformation of reciprocal radii it is sufficient to replace ρ by $\frac{R^2}{\rho}$. Thus the first term of the previous expression becomes

$$- R^2 \frac{R^2 - l \rho \cos \gamma}{(l^2 \rho^2 - R^4 - 2 l R^2 \rho \cos \gamma)^{\frac{1}{2}}}.$$

The second term equals

$$\frac{\rho l (l \rho - R^2 \cos \gamma)}{(R^4 + l^2 \rho^2 - 2 l R^2 \rho \cos \gamma)^{\frac{1}{2}}}$$

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It is found then that

$$a_2 = -\lambda \frac{l^2 \rho^2 - R^4}{(R^4 + l^2 \rho^2 - 2 l \rho R^2 \cos \gamma)^{\frac{3}{2}}}.$$

In calculating $a_4, a_6 \dots$ there are no more difficulties and

$$\begin{aligned} a_4 &= -\lambda^2 \frac{\rho}{R} \frac{\partial}{\partial \rho} \left[\frac{R^4 - l^2 \rho^2}{(R^4 + l^2 \rho^2 - 2 l \rho R^2 \cos \gamma)^{\frac{3}{2}}} \right] \\ &= -\frac{\lambda^2}{R} \frac{\partial}{\partial \log \rho} \left[\frac{R^4 - l^2 \rho^2}{(R^4 + l^2 \rho^2 - 2 l \rho R^2 \cos \gamma)^{\frac{3}{2}}} \right]. \end{aligned}$$

In general,

$$a_{2n} = (-1)^{n-1} \frac{\lambda^n}{R^{n-1}} \frac{\partial^{n-1}}{\partial (\log \rho)^{n-1}} \left[\frac{R^4 - l^2 \rho^2}{(R^4 + l^2 \rho^2 - 2 l \rho R^2 \cos \gamma)^{\frac{3}{2}}} \right].$$

Consequently,

$$\begin{aligned} \Psi &= a_0 + \sum_1^{\infty} (-1)^{n-1} \frac{\lambda^n}{R^{n-1}} \frac{\partial^{n-1}}{\partial (\log \rho)^{n-1}} \\ &\quad \left[\frac{R^4 - l^2 \rho^2}{(R^4 + l^2 \rho^2 - 2 l \rho R^2 \cos \gamma)^{\frac{3}{2}}} \right] \frac{(t_1 - t)^{2n}}{2 n !}. \end{aligned}$$

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= - \sum_1^{\infty} (-1)^{n-1} \frac{\lambda^n}{R^{n-1}} \frac{\partial^{n-1}}{\partial (\log \rho)^{n-1}} \\ &\quad \left[\frac{R^4 - l^2 \rho^2}{(R^4 + l^2 \rho^2 - 2 l \rho R^2 \cos \gamma)^{\frac{3}{2}}} \right] \frac{(t_1 - t)^{2n-1}}{(2 n - 1) !}. \end{aligned}$$

In order to calculate the formula (D) of section 4 it is necessary to evaluate ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$, that is to say, to put $t = 0$ in the previous series. Further it is the values at the surface of the sphere which have to be found. Finally, this expression must be derived with respect to t_1 .

Let us then adopt polar coördinates and put

$$\begin{aligned} x &= \rho \sin \theta \cos \phi, & y &= \rho \sin \theta \sin \phi, & z &= \rho \cos \theta, \\ x_0 &= l \sin \theta_0 \cos \phi_0, & y_0 &= l \sin \theta_0 \sin \phi_0, & z &= l \cos \theta_0. \end{aligned}$$

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Then $\cos \gamma = \cos \phi \cos \phi_0 + \sin \phi \sin \phi_0 \cos (\theta - \theta_0)$.

Let us write

$$\Theta(l, \theta_0, \phi_0, \theta, \phi, t) = \sum_1^{\infty} (-1)^{n-1} \frac{\lambda^{n-1}}{R^{n-1}} \frac{\partial^{n-1}}{\partial (\log l)^{n-1}} \left[\frac{R^2 - l^2}{R^2 + l^2 - 2 R l \cos \gamma} \right]^{\frac{1}{2}} \frac{t^{2n-1}}{(2n-1)!}.$$

Formula (D) can be written

$$(D_a) \Phi(l, \theta_0, \phi_0, t) = \frac{R}{\gamma \pi} \int_{\omega} \phi'_0(\theta, \phi) \Theta(l, \theta_0, \phi_0, \theta, \phi, t) \sin \theta d\theta d\phi \\ + \frac{R}{\gamma \pi} \frac{d}{dt} \int_{\omega} \phi_0(\theta, \phi) \Theta(l, \theta_0, \phi_0, \theta, \phi, t) \sin \theta d\theta d\phi,$$

where for simplification we have written

$$\Phi_0(\theta, \phi) = \phi_0(R, \theta, \phi, t), t = 0 \\ \Phi'_0(\theta, \phi) = \left\{ \frac{d}{dt} \phi_0(R, \theta, \phi, t) \right\}, t = 0.$$

The formula we have been seeking to find is the general formula in the case of the sphere.

If, instead of a sphere, the liquid occupies a hemisphere and the diametral plane constitutes the rigid boundary so that the curved surface is free, the method of images will provide the solution in a similar manner. The same holds in the case where the liquid occupies a section of a sphere between two rigid diametral planes the angle between which equals $\frac{\pi}{n}$, where n is an integer.

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